April 20, 2017 Math 124a Name Key

1. (3pts) Evaluate the following integral:

$u = 3 - x^3 \qquad du = -3x^2 dx$

$$\int x^2 e^{3-x^3}\, dx.$$

$= -\frac{1}{3}\int -3x e^{3-x^3} dx = -\frac{1}{3}\int e^u du = -\frac{1}{3} e^u + c$

Integral = $-\frac{1}{3} e^{3-x^3} + C$

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2. (9pts) Solve the following initial value problem:

a. $\dfrac{dy}{dt} = -6ty^{2/3}, \qquad y(0) = 27,$

$\int y^{-2/3} dy = -6\int t\, dt$

$3y^{1/3} = -3t^2 + c$

$y(t) = \left(\frac{c}{3} - t^2\right)^3$

$y(0) = 27 = \left(\frac{c}{3}\right)^3 \Rightarrow \frac{c}{3} = 3$

$y(t) =$ $(3 - t^2)^3$

b. $\dfrac{dy}{dt} = 5 - \dfrac{10}{t^2}, \qquad y(1) = 3.$

$y(t) = \int (5 - 10t^{-2}) dt = 5t + 10t^{-1} + c$

$y(1) = 3 = 5 + 10 + c$

$c = -12$

$y(t) =$ $5t + 10t^{-1} - 12$

c. $\dfrac{dy}{dt} = \dfrac{4ty^2}{\sqrt{t^2+16}}, \qquad y(0) = 1.$ $\quad u = t^2 + 16 \quad du = 2t\,dt$

$\int y^{-2} dy = \int \frac{4t\, dt}{(t^2+16)^{1/2}} \Rightarrow -\frac{1}{y(t)} = 2\int u^{-1/2} du = 4u^{1/2} + c$

$y(t) = -\dfrac{1}{4\sqrt{t^2+16} + c}$

$y(t) =$ $\dfrac{1}{17 - (t^2+16)^{1/2}}$

$y(0) = 1 = -\dfrac{1}{16 + c}$

$\Rightarrow c = -17$

3. (8pts) A number of countries around the world are having a dramatic decline in their growth rate to the point where their populations will actually begin to decline early in this century. Consider the case of United Kingdom. Its population was 50.62 million in 1950, 55.66 million in 1970, and 57.25 million in 1990. Use the nonautonomous Malthusian growth model given by

$$\frac{dP}{dt} = (a - bt)P, \qquad P(0) = 50.62.$$

Let t be the number of years after 1950, then solve this differential equation. Use the data at $t = 0$, 20, and 40 years to find the constants a and b. When does the model predict that United Kingdom will have its largest population (value of t in years after 1950) and what is that population?

$\int \frac{dP}{P} = \int (a - bt)dt \Rightarrow \ln|P| = at - \frac{bt^2}{2} + c$

$c = \ln(50.62)$

$P(t) = 50.62e^{at - bt^2/2}$

$-2(20a - 200b = \ln\left(\frac{55.66}{50.62}\right) = 0.094915)$ $\quad -400b = 0.12308 - 0.189830$

$40a - 800b = \ln\left(\frac{57.25}{50.62}\right) = 0.12308$ $\quad a = 10b + \frac{0.094915}{20}$

$a =$ 0.0064145 $\quad b =$ 0.00016687

$P(t) =$ $50.62e^{0.0064145t - 0.000083436t^2}$ (million)

$t_{max} =$ 38.439 $\quad P(t_{max}) =$ 57.262 (million)

$t_{max} = a/b$