1. a. The equilibria are found by letting $P_{n+1} = P_n = P_e$, so

$$P_e = P_e + 1.14P_e\left(1 - \frac{P_e}{370}\right) \qquad \text{or} \qquad 1.14P_e\left(1 - \frac{P_e}{370}\right) = 0.$$

It is easy to see that $P_e = 0$ and 370.

If we write $F(P) = 2.14\,P - \frac{1.14}{370}P^2$, then the derivative is

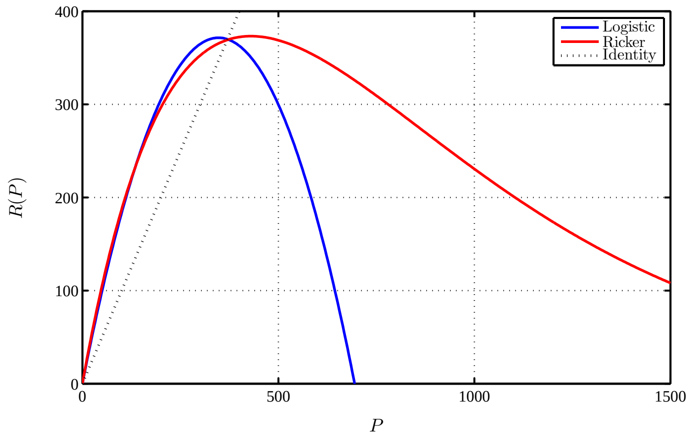
$$F'(P) = 2.14 - \frac{2.28}{370}P = 2.14 - 0.006162\,P.$$

At $P_e = 0$, we see $F'(0) = 2.14$, so it follows that this equilibrium is unstable with solutions monotonically moving away from $P_e = 0$. At $P_e = 370$, we see $F'(370) = -0.14$, so it follows that this equilibrium is stable with solutions oscillating and moving toward $P_e = 370$.

b. Given the updating function $R(P) = 2.36\,Pe^{-P/430}$, we find the derivative

$$R'(P) = 2.36\left(Pe^{-P/430}\left(-\frac{1}{430}\right) + e^{-P/430}\right) = 2.36\,e^{-P/430}\left(1 - \frac{P}{430}\right).$$

The derivative is zero when $P = 430$. So there is a maximum at $P_{max} = 430$ with $R(P_{max}) = 2.36(430)e^{-1} \approx 373.32$. There is a horizontal asymptote to the right with $\lim_{P\to\infty} R(P) = 0$. The graph is below. Note the similarity of the logistic model from Part a to the Ricker's model from zero to the carrying capacity.



Problem 1b

c. The equilibria are found by solving $P_e = 2.36\,P_e e^{-P_e/430}$. Thus, one equilibrium is $P_e = 0$. By dividing out P_e, we find:

$$1 = 2.36\,e^{-P_e/430} \quad \text{or} \quad e^{P_e/430} = 2.36 \quad \text{or} \quad P_e = 430\ln(2.36) \approx 369.224.$$

At $P_e = 0$, we see $R'(0) = 2.36$, so it follows that this equilibrium is unstable with solutions monotonically moving away from $P_e = 0$. At $P_e = 369.22$, we see $R'(430\ln(2.36)) = 1 - \ln(2.36) = 0.1413$, so it follows that this equilibrium is stable with solutions monotonically approaching $P_e = 369.22$.

2. a. The equilibria are found by letting $P_{n+1} = P_n = P_e$, so

$$P_e = 2.54P_e - 0.01P_e^2 \qquad \text{or} \qquad 0.01P_e\,(154 - P_e) = 0.$$

It is easy to see that $P_e = 0$ and 154.

Since $F(P) = 2.54P - 0.01P^2$, then the derivative is

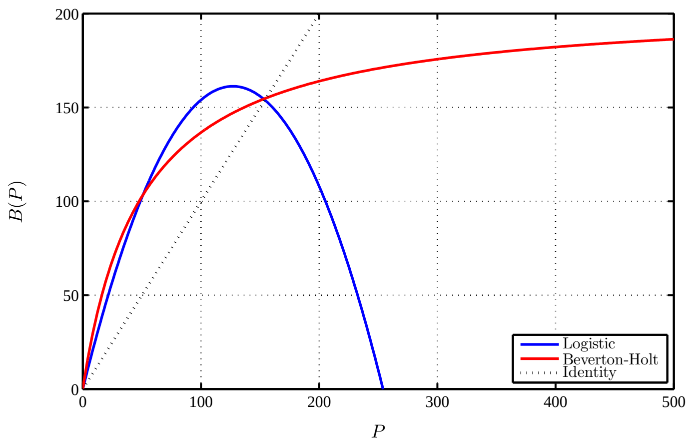
$$F'(P) = 2.54 - 0.02\,P.$$

At $P_e = 0$, we see $F'(0) = 2.54$, so it follows that this equilibrium is unstable with solutions monotonically moving away from $P_e = 0$. At $P_e = 154$, we see $F'(154) = -0.54$, so it follows that this equilibrium is stable with solutions oscillating and moving toward $P_e = 154$.

b. Given the updating function $B(P) = \frac{4.1P}{1+0.02P}$, we find the derivative

$$B'(P) = 4.1\frac{((1+0.02P)\cdot 1 - 0.02P)}{(1+0.02P)^2} = \frac{4.1}{(1+0.02P)^2}.$$

The derivative satisfies $B'(P) > 0$ for all $P \geq 0$, so this function is always increasing (no maxima or minima). There is a horizontal asymptote at $B = 205$. The graph is below. Note that the logistic model from Part a tracks a similar path to the Beverton-Holt model from zero to the carrying capacity.



Problem 2b

c. The equilibria are found by solving $P_e = \frac{4.1P_e}{1+0.02P_e}$. Thus, one equilibrium is $P_e = 0$. By dividing out P_e, we find:

$$1 = \frac{4.1}{1+0.02P_e} \quad \text{or} \quad 1 + 0.02P_e = 4.1 \quad \text{or} \quad P_e = 155.$$

At $P_e = 0$, we see $B'(0) = 4.1$, so it follows that this equilibrium is unstable with solutions monotonically moving away from $P_e = 0$. At $P_e = 155$, we see $B'(155)) = \frac{4.1}{(1+0.02(155))^2} = 0.2439$, so it follows that this equilibrium is stable with solutions monotonically approaching $P_e = 155$.

3. a. Rewrite the second integral as a power, then

$$\begin{aligned} \int \left(6\cos(3x) - \frac{2}{x^3}\right) dx &= 6\int \cos(3x)dx - 2\int x^{-3}dx \\ &= 6\frac{\sin(3x)}{3} - 2\frac{x^{-2}}{-2} + C = 2\sin(3x) + \frac{1}{x^2} + C \end{aligned}$$

b. Let $u = x^2 + 4\,x - 5$, so $du = (2\,x + 4)\,dx = 2(x+2)\,dx$.

$$\begin{aligned}\int \left(x^2+4\,x-5\right)^3 (x+2)dx &= \frac{1}{2}\int \left(x^2+4\,x-5\right)^3 2(x+2)dx \\ &= \frac{1}{2}\int u^3 du \\ &= \frac{1}{8}u^4 + C \;=\; \frac{1}{8}(x^2+4\,x-5)^4 + C\end{aligned}$$

c. The first integral is written as a power, while the second integral uses the substitution $u = 3\,x - 2$, so $du = 3\,dx$.

$$\begin{aligned}\int \left(3\,x^{-2} + 3\cos(3\,x-2)\right) dx &= 3\int x^{-2}dx + \int \cos(u)du \\ &= 3\frac{x^{-1}}{-1} + \sin(u) + C = -\frac{3}{x} + \sin(3\,x-2) + C\end{aligned}$$

d. Let $u = -x^2$, so $du = -2\,x\,dx$.

$$\begin{aligned}\int \left(2x\,e^{-x^2} - 4\,x\right) dx &= -\int e^u\,du - 4\frac{x^2}{2} \\ &= -e^u - 2\,x^2 + C = -e^{-x^2} - 2\,x^2 + C\end{aligned}$$

e. Rewrite the second integral as a power, then

$$\begin{aligned}\int \left(4e^{-2x} + \frac{3}{\sqrt{x}}\right) dx &= -2e^{-2x} + 3\int x^{-1/2}dx \\ &= -2e^{-2x} + 6\sqrt{x} + C\end{aligned}$$

f. Let $u = \sin(4x)$, so $du = 4\cos(4x), dx$.

$$\begin{aligned}\int \left(\frac{7}{x} + 8\sin^3(4x)\cos(4x)\right) dx &= 7\ln(x) + 2\int u^3\,du \\ &= 7\ln(x) + \frac{u^4}{2} + C \;=\; 7\ln(x) + \frac{\sin^4(4x)}{2} + C\end{aligned}$$

4. a. Both antiderivatives are standard for this problem

$$\begin{aligned}\int_0^{2\pi} (\cos(t/4) + t)\,dt &= \left(4\sin(t/4) + \frac{t^2}{2}\right)\Bigg|_{t=0}^{2\pi} \\ &= 4\sin(\pi/2) + \frac{4\pi^2}{2} - 4\sin(0) - 0 = 4 + 2\,\pi^2\end{aligned}$$

b. The first integral uses the substitution $u = x^2 + 1$, so $du = 2\,x\,dx$ with endpoints $u = 2$ when $x = 1$ and $u = 5$ when $x = 2$, while the second antiderivative is the natural logarithm

$$\begin{aligned}\int_1^2 \left(\frac{6x}{x^2+1} + \frac{2}{x}\right) dx &= 3\int_2^5 \frac{du}{u} + 2\ln|x|\Big|_{x=1}^2 \\ &= 3\ln|u|\Big|_{u=2}^5 + 2\,\ln(2) - 2\,\ln(1) \\ &= 3\ln(5) - 3\ln(2) + 2\,\ln(2) = 3\ln(5) - \ln(2) = \ln(62.5) \approx 4.135\end{aligned}$$

c. Let $u = 25 - x^2$, so $du = -2x\,dx$ with endpoints $u = 25$ when $x = 0$ and $u = 0$ when $x = 5$.

$$\begin{aligned}\int_0^5 x\sqrt{25 - x^2}\,dx &= -\frac{1}{2}\int_{25}^0 u^{\frac{1}{2}}\,du \\ &= -\frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_{u=25}^0 \\ &= -\frac{1}{3}\left(0^{\frac{3}{2}} - 25^{\frac{3}{2}}\right) = \frac{125}{3}\end{aligned}$$

d. For the first integral, let $u = \ln(x)$, so $du = dx/x$ with endpoints $u = 0$ when $x = 1$ and $u = \ln(4)$ when $x = 4$. The second integral is a power rule.

$$\begin{aligned}\int_1^4 \left(\frac{\ln(x)}{x} + \frac{3}{\sqrt{x}}\right) dx &= \int_0^{\ln(4)} u\,du + 3\int_1^4 x^{-\frac{1}{2}}\,dx \\ &= \frac{u^2}{2}\Bigg|_{u=0}^{\ln(4)} + 6\,x^{\frac{1}{2}}\Big|_1^4 \\ &= \frac{1}{2}(\ln(4))^2 + 6(2-1) = 6 + 2\,(\ln(2))^2\end{aligned}$$

5. a. This is a time varying differential equation. It can be written

$$y(t) = \int \left(1 + e^{-t}\right) dt = t - e^{-t} + C.$$

The initial condition $y(0) = 3 = -1+C$, which implies $C = 4$. Hence, the solution is $y(t) = t - e^{-t} + 4$.

b. This is a time varying differential equation. It can be written

$$y(t) = \int \left(2 - \frac{4}{t}\right) dt = 2\,t - 4\,\ln(t) + C.$$

The initial condition $y(1) = 5 = 2+C$, which implies $C = 3$. Hence, the solution is $y(t) = 2\,t - 4\,\ln(t) + 3$.

c. This is a separable differential equation. It can be written

$$\int 2y\,dy = \int 3t^2\,dt \qquad \text{or} \qquad y^2 = t^3 + C.$$

It follows that $y(t) = \pm\sqrt{t^3 + C}$. The initial condition $y(0) = 4 = \sqrt{C}$, which implies $C = 16$. Hence, the solution is

$$y(t) = \sqrt{t^3 + 16}.$$

d. This is a linear differential equation, which can be written

$$\frac{dy}{dt} = -0.02(y - 100).$$

With the substitution $z(t) = y(t) - 100$, we have

$$\frac{dz}{dt} = -0.02z, \qquad z(0) = y(0) - 100 = -95.$$

Thus, $z(t) = -95\,e^{-0.02\,t}$. It follows that

$$y(t) = 100 - 95\,e^{-0.02\,t}.$$

e. This is a separable differential equation. It can be written

$$\int \frac{dy}{y} = \int \frac{2\,t\,dt}{t^2 + 1}.$$

The right integral uses the substitution $u = t^2 + 1$, so $du = 2\,t\,dt$. Hence,

$$\begin{aligned} \ln|y(t)| &= \int \frac{du}{u} = \ln|u| + C = \ln(t^2 + 1) + C \\ y(t) &= e^{\ln(t^2+1)+C} = A(t^2 + 1), \end{aligned}$$

where $A = e^C$. The initial condition $y(0) = 3 = A$, which implies $A = 3$. Hence, the solution is

$$y(t) = 3(t^2 + 1).$$

f. This is a separable differential equation. It can be written

$$\int \frac{dy}{y} = \int (2 - 0.2t)dt \qquad \text{or} \qquad \ln|y| = 2\,t - 0.1\,t^2 + C.$$

It follows that $y(t) = e^{2\,t - 0.1\,t^2 + C} = A\,e^{2\,t - 0.1\,t^2}$ with $A = e^C$. The initial condition $y(0) = 10 = A$, which implies $A = 10$. Hence, the solution is

$$y(t) = 10\,e^{2\,t - 0.1\,t^2}.$$

g. This is a time-varying differential equation, so we integrate giving

$$y(t) = \int (4 - 2\,\sin(2(t-3)))dt = 4\,t - 2\int \sin(2(t-3))dt.$$

With the substitution $u = 2(t-3)$ and $du = 2\,dt$, we have

$$y(t) = 4\,t - \int \sin(u)du = 4\,t + \cos(u) + C = 4\,t + \cos(2(t-3)) + C.$$

With the initial condition $y(3) = 5$, $12 + \cos(0) + C = 5$ or $C = -8$. It follows that

$$y(t) = 4\,t + \cos(2(t-3)) - 8.$$

h. This is a separable differential equation. It can be written

$$\int e^y dy = \int e^t dt \qquad \text{or} \qquad e^y = e^t + C.$$

It follows that $y(t) = \ln(e^t + C)$. The initial condition $y(0) = 6 = \ln(1 + C)$, which implies $C = e^6 - 1$. Hence, the solution is

$$y(t) = \ln(e^t + e^6 - 1).$$

6. a. The solution with the population in millions is given by

$$P(t) = 50.2e^{rt},$$

where t is in years after 1880. From the population in 1890, we have $62.9 = 50.2e^{10r}$ or $e^{10r} = 1.2530$. Thus, $r = 0.02255$. To find the time until the population doubles, we compute $100.4 = 50.2e^{rt}$ or $t = \ln(2)/r \approx 30.7$. This suggests that the population of the U.S. doubles from 1880 around 1911, assuming that the rate of growth stays constant.

b. The model predicts that the population in 1900 is

$$P(20) = 50.2e^{20r} \approx 78.8.$$

The error between the model and the actual population is

$$100\frac{(P(20) - 76.0)}{76.0} = 100\frac{(78.8 - 76.0)}{76.0} = 3.7\%.$$

7. a. The solution to the radioactive decay problem is

$$R(t) = 30e^{-kt}.$$

With the half-life of 8 years, $R(8) = 15 = 30e^{-8k}$ or $e^{8k} = 2$. Thus,

$$8k = \ln(2) \qquad \text{or} \qquad k = \frac{\ln(2)}{8} \approx 0.08664.$$

After 3 days,

$$R(3) = 30e^{-3k} \approx 23.13 \text{ mCi}.$$

b. The length of time for the original 30 mCi of ^{131}I to decay to 5 mCi of ^{131}I satisfies $R(t) = 5 = 30e^{-kt}$ or $e^{kt} = 6$. It follows that $kt = \ln(6)$ or

$$t = \frac{\ln(6)}{k} \approx \frac{\ln(6)}{0.08664} \approx 20.68 \text{ days}.$$

c. The (maximum) cumulative exposure with $k = \frac{\ln(2)}{8}$, which would include the patient and the waste products, satisfies:

$$30\int_0^{20} e^{-kt}dt = -\frac{30}{k}e^{-kt}\Big|_0^{20} = \frac{240}{\ln(2)}\left(1 - e^{-5\ln(2)/2}\right) = 285.04 \text{ mCi} \cdot \text{day}.$$

8. a. The solution to the white lead problem is $P(t) = 10e^{-kt}$, where $t = 0$ represents 1970. From the data at 1975, we have $8.5 = 10e^{-5k}$ or $e^{5k} = 10/8.5 = 1.17647$. Thus, $k = 0.032504$ yr^{-1}. To find the half-life, we compute $5 = 10e^{-kt}$, so $t = \ln(2)/k = 21.33$ yr is the half-life of lead-210.

b. The differential equation can be written $P' = -k(P - r/k)$, so we make the substitution $z(t) = P(t) - r/k$. This leaves the initial value problem

$$z' = -kz, \quad z(0) = P(0) - r/k = 10 - r/k,$$

which has the solution $z(t) = (P(0) - r/k)e^{-kt} = P(t) - r/k$. Thus, the solution is

$$P(t) = \left(10 - \frac{r}{k}\right)e^{-kt} + \frac{r}{k} = 2.3086e^{-kt} + 7.6914,$$

where $k = 0.032504$. In the limit,

$$\lim_{t\to\infty} P(t) = 7.6914 \text{ disintegrations per minute of } {}^{210}\text{Pb}.$$

9. a. The differential equation describing the temperature of the tea satisfies

$$H' = -k(H - 21), \quad H(0) = 85 \text{ and } H(5) = 81.$$

Make the substitution $z(t) = H(t) - 21$, which gives the differential equation

$$z' = -kz, \quad z(0) = H(0) - 21 = 64.$$

The solution becomes $z(t) = 64e^{-kt} = H(t) - 21$ or

$$H(t) = 64e^{-kt} + 21.$$

To find k, we solve $H(5) = 81 = 64e^{-5k} + 21$ or $e^{5k} = 64/60 = 1.0667$. Thus, $k = 0.012908$ min^{-1}. The water was at boiling point when $64e^{-kt} + 21 = 100$ or $e^{-kt} = 79/64$. It follows that $t = -\ln(79/64)/k = -16.3$ min. This means that the talk went 16.3 min over its scheduled ending.

b. To obtain a temperature of at least 93°C, then we need to find the time that satisfies $H(t) = 93 = 64e^{-kt} + 21$, so $e^{-kt} = 72/64 = 1.125$. Solving for t gives $t = -\ln(72/64)/k = -9.125$ min. It follows that you must arrive at the hot water within $16.3 - 9.1 = 7.2$ min of the scheduled end of the talks.

10. a. Substituting the parameters into the differential equation gives

$$c' = \frac{1}{10^6}(22000 - 2000c) = -0.002(c - 11).$$

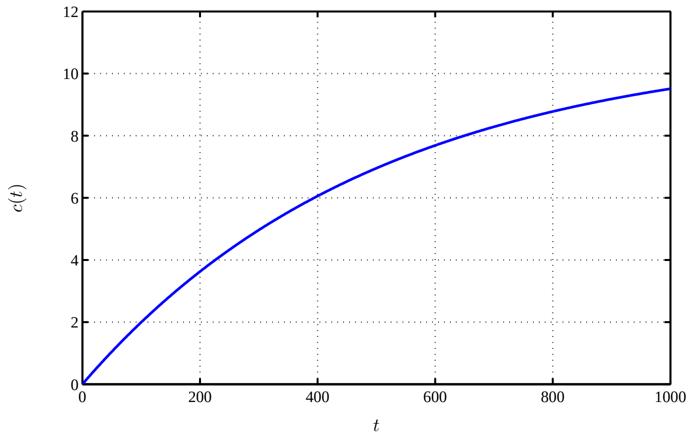
We make the substitution $z(t) = c(t) - 11$, which gives the initial value problem $z' = -0.002z$ with $z(0) = c(0) - 11 = -11$. The solution of this differential equation is $z(t) = -11e^{-0.002t} = c(t) - 11$, so

$$c(t) = 11 - 11e^{-0.002t}.$$

b. Solve the equation $c(t) = 11 - 11e^{-0.002t} = 5$, so $e^{0.002t} = 11/6$ or $t = 500\ln(11/6) = 303.1$ days. The limiting concentration

$$\lim_{t\to\infty} c(t) = 11.$$

The graph is below.



Problem 10

11. Integrating the acceleration due to gravity, we see that the velocity is given by $v(t) = v_0 - 32\,t$. Similarly, the height is the integral of the velocity, so $h(t) = \int(v_0 - 32\,t)dt = -16\,t^2 + v_0 t$, where the integration constant is zero, since the initial height is zero. The maximum height occurs when the velocity is zero, so $t_{max} = v_0/32$. But

$$h(v_0/32) = \frac{v_0^2}{32} - \frac{v_0^2}{64} = \frac{v_0^2}{64} = 8.$$

It follows that $v_0^2 = 512$ or $v_0 = 16\sqrt{2}$, which is the initial upward velocity. The length of time that the kangaroo stays in the air is twice the length of time to reach the maximum, so it stays in the air for $t_{hang} = \sqrt{2}$ sec.

12. The differential equation is separable, so write

$$\int T^{-\frac{1}{2}} dT = k \int dt \qquad \text{or} \qquad 2\,T^{\frac{1}{2}}(t) = kt + C.$$

It follows that

$$T(t) = \left(\frac{kt + C}{2}\right)^2.$$

The initial condition $T(0) = 1$ implies $C = 2$, so $T(t) = \left(\frac{kt}{2} + 1\right)^2$. Since $T(4) = \left(\frac{4\,k}{2} + 1\right)^2 = 25$, $2\,k + 1 = 5$ or $k = 2$. Thus, the solution for the spread of the disease in this orchard is

$$T(t) = (t + 1)^2.$$

When $t = 10$, $T(10) = 121$.

13. a. The solution of the Malthusian growth equation for Japan is $J(t) = 116.8e^{rt}$ (in millions) from the differential equation and the population in 1980 with $t = 0$ corresponding to 1980. Since the population in 1990 is 123.5 (million), we have $J(10) = 123.5 = 116.8e^{10r}$. Thus,

$$e^{10r} = \frac{123.5}{116.8} \approx 1.0574 \quad \text{or} \quad 10\,r = \ln(1.0574) \approx 0.055778 \quad \text{or} \quad r = 0.005578.$$

The doubling time is computed by solving $J(t) = 233.6 = 116.8e^{rt}$, so

$$e^{rt} = 2 \quad \text{or} \quad rt = \ln(2) \approx 0.69315 \quad \text{or} \quad t = \frac{\ln(2)}{r} \approx 124.3 \text{ yr}$$

b. The differential equation for Bangladesh is given by

$$\frac{dB}{dt} = kB, \quad B(0) = 88.1.$$

A similar calculation is used for Bangladesh, so $B(t) = 88.1e^{kt}$ with $B(10) = 110.1 = 88.1e^{kt}$. Solving for the growth constant k as above, we find

$$k = \frac{1}{10} \ln\left(\frac{110.1}{88.1}\right) \approx 0.02229.$$

The population in 2000 is found by evaluating

$$B(20) = 88.1e^{20k} \approx 137.6 \text{ million}.$$

c. The populations of Japan and Bangladesh are equal when $B(t) = J(t)$, so

$$88.1e^{kt} = 116.8e^{rt} \quad \text{or} \quad \frac{e^{kt}}{e^{rt}} = \frac{116.8}{88.1} \approx 1.3258 \quad \text{or} \quad e^{(k-r)t} = e^{0.016714t} = 1.3258$$

$$0.016714t = \ln(1.3258) \approx 0.28199 \quad \text{or} \quad t = \frac{0.28199}{0.016714} \approx 16.87 \text{ years}$$

It follows that these models predict that the population of Bangladesh exceeded the population of Japan in 1997.

14. The differential equation with the information in the problem is given by:

$$\frac{dH}{dt} = -k(H - 25), \qquad H(0) = 35,$$

where $t = 0$ is 7 AM. We make the change of variables $z(t) = H(t) - 25$, so $z(0) = 10$. The problem now becomes

$$\frac{dz}{dt} = -kz, \qquad z(0) = 10,$$

which has the solution

$$z(t) = 10\,e^{-kt} \quad \text{or} \quad H(t) = 25 + 10\,e^{-kt}.$$

From the information at 9 AM, we see

$$H(2) = 33.5 = 25 + 10\,e^{-2k} \quad \text{or} \quad e^{2k} = \frac{10}{8.5} \quad \text{or} \quad k = \frac{\ln\left(\frac{10}{8.5}\right)}{2} = 0.081259.$$

It follows that

$$H(t) = 25 + 10\,e^{-0.081259t}.$$

The time of death is found by solving

$$H(t_d) = 39 = 25 + 10\,e^{-0.081259t_d} \quad \text{or} \quad e^{-0.081259t_d} = \frac{14}{10} \quad \text{or} \quad t_d = -\frac{\ln(1.4)}{0.081259} = -4.1407.$$

It follows that the time of death is 4 hours and 8.4 min before the body is found, which gives the time of death around 2:52 AM.

15. a. Let $A(t)$ be the amount of drug in the body, then the concentration of the drug is given by $c(t) = A(t)/10$. We first write the differential equation for the change in amount of drug in the body

$$\frac{dA}{dt} = \textit{amt entering} - \textit{amt leaving} = 1(0.2) - 1 \cdot c.$$

The differential equation for the concentration of drug satisfies

$$\frac{dc}{dt} = 0.02 - 0.1c = -0.1(c - 0.2), \qquad c(0) = 0.$$

Let $z(t) = c(t) - 0.2$, then we transform the linear differential equation above into

$$\frac{dz}{dt} = -0.1z, \qquad z(0) = -0.2.$$

which has the solution

$$z(t) = -0.2\,e^{-0.1t} \quad \text{or} \quad c(t) = 0.2 - 0.2\,e^{-0.1t}.$$

b. The tumor responds when $c(t) = 0.1$, solving $c(t) = 0.1 = 0.2 - 0.2\,e^{-0.1t}$ or $e^{0.1t} = 2$. It easily follows that the time for a response to begin is $t = 10\ln(2) = 6.9315$ days.

c. If the body metabolizes 0.05 μg/day, then the new equation for the amount of drug in the body is

$$\frac{dA}{dt} = 1(0.2) - 0.05 - 1 \cdot c = 0.15 - c.$$

The differential equation for the concentration of drug satisfies

$$\frac{dc}{dt} = 0.015 - 0.1c = -0.1(c - 0.15), \qquad c(0) = 0.$$

The limiting concentration is reached when $\frac{dc}{dt} = 0$. Substituting this in the differential equation above, we see

$$0 = -0.1(c - 0.15) \quad \text{or} \quad c = 0.15.$$

It follows that

$$\lim_{t\to\infty} c(t) = 0.15\ \mu\text{g/l}.$$

16. a. The solution of the Malthusian growth model is $B(t) = 1000\,e^{0.01\,t}$. The population doubles when the bacteria reaches 2000, so $1000\,e^{0.01\,t} = 2000$ or $e^{0.01\,t} = 2$. Thus, $0.01\,t = \ln(2)$ or $t = 100\ln(2) \approx 69.3$ min for the population to double.

b. The model with time-varying growth is a separable differential equation, so

$$\frac{dB}{dt} = 0.01(1 - e^{-t})\,B \quad \text{or} \quad \int \frac{dB}{B} = 0.01 \int (1 - e^{-t})dt$$

$$\ln|B(t)| = 0.01(t + e^{-t}) + C \quad \text{or} \quad B(t) = A\,e^{0.01(t+e^{-t})},$$

where $A = e^C$. With the initial condition, $B(0) = 1000 = A\,e^{0.01}$ or $A = 1000\,e^{-0.01}$. Thus, the solution to this time-varying growth model is

$$B(t) = 1000\,e^{0.01(t+e^{-t}-1)}.$$

c. The Malthusian growth model gives $B(5) = 1051$ and $B(60) = 1822$, while the modified growth model gives $B(5) = 1041$ and $B(60) = 1804$.

17. a. The solution to the Malthusian growth model is given by $P(t) = 100\,e^{0.2\,t}$. This population doubles when $100\,e^{0.2\,t} = 200$ or $e^{0.2\,t} = 2$, so $t = 5\ln(2) \approx 3.466$ yrs.

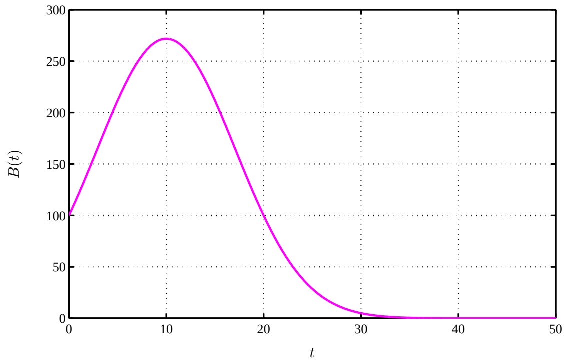
b. This model, including the modification for habitat encroachment, is a separable differential equation. It can be written

$$\int \frac{dP}{P} = \int (0.2 - 0.02t)dt \qquad \text{or} \qquad \ln|P| = 0.2\,t - 0.01\,t^2 + C.$$

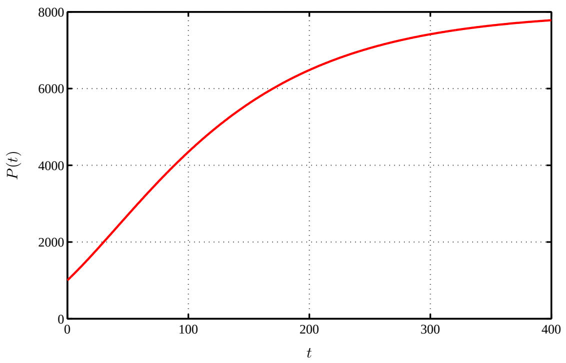
It follows that $P(t) = e^{0.2\,t - 0.01\,t^2 + C} = Ae^{0.2\,t - 0.01\,t^2}$, where $A = e^C$. The initial condition $P(0) = 100 = A$, which implies $A = 100$. Hence, the solution satisfies

$$P(t) = 100\,e^{0.2\,t - 0.01\,t^2}.$$

c. We examine the differential equation in Part b and see that $\frac{dP}{dt} = 0$ when $0.2 - 0.02t = 0$, which implies that $t = 10$. Thus, the maximum of population is $P(10) = 100\,e \approx 271.8$. If we solve $P(t) = 100\,e^{0.2\,t-0.01\,t^2} = 100$, then this is equivalent to $e^{0.2\,t-0.01\,t^2} = 1$ or $0.2\,t - 0.01\,t^2 = -0.01\,t(t - 20) = 0$. Thus, either $t = 20$ (or 0), so the population returns to 100 after 20 years. The graph of the population can be seen below.



Problem 17



Problem 18

18. a. This population of cells in a declining medium satisfies a separable differential equation, which can be written

$$\int P^{-2/3} dP = \int 0.3\, e^{-0.01t} dt \qquad \text{or} \qquad 3\, P^{1/3}(t) = -30\, e^{-0.01t} + 3C.$$

It follows that $P^{1/3}(t) = -10\, e^{-0.01t} + C$, so $P(t) = \left(C - 10\, e^{-0.01t}\right)^3$. The initial condition $P(0) = 1000 = (C - 10)^3$, which implies $C = 20$. The solution is given by

$$P(t) = \left(20 - 10e^{-0.01\, t}\right)^3.$$

b. This population doubles when $P(t) = (20 - 10e^{-0.01\, t})^3 = 2000$, so $20 - 10e^{-0.01\, t} = 10\sqrt[3]{2}$ or $e^{-0.01\, t} = 2 - \sqrt[3]{2}$. It follows that $t = 100 \ln\left(\frac{1}{2-\sqrt[3]{2}}\right) \approx 30.1$ hr. For large t, $\lim_{t\to\infty} e^{-0.01\, t} = 0$, so $\lim_{t\to\infty} P(t) = 20^3 = 8000$. Thus, there is a horizontal asymptote at $P = 8000$, so the population tends towards this value. The graph of the population can be seen above.

19. a. The change in amount of phosphate, $P(t)$, is found by adding the amount entering and subtracting the amount leaving.

$$\frac{dP}{dt} = 200 \cdot 10 - 200 \cdot c(t),$$

where $c(t)$ is the concentration in the lake with $c(t) = P(t)/10,000$. By dividing the equation by the volume, the concentration equation is given by

$$\frac{dc}{dt} = 0.2 - 0.02c = -0.02(c - 10), \qquad c(0) = 0.$$

With the substitution $z(t) = c(t) - 10$, the equation above reduces to the problem

$$\frac{dz}{dt} = -0.02z, \qquad z(0) = -10,$$

which has the solution $z(t) = -10\, e^{-0.02\, t}$. Thus, the concentration is given by

$$c(t) = 10 - 10\, e^{-0.02\, t}.$$

b. The differential equation describing the growth of the algae is given by

$$\frac{dA}{dt} = 0.5(1 - e^{-0.02\, t})A^{2/3}.$$

By separating variables, we see

$$\begin{aligned}
\int A^{-2/3} dA &= 0.5 \int (1 - e^{-0.02\,t})dt \\
3\,A^{1/3}(t) &= 0.5(t + 50\,e^{-0.02\,t}) + C \\
A(t) &= \left(\frac{0.5(t + 50\,e^{-0.02\,t}) + C}{3}\right)^3
\end{aligned}$$

From the initial condition $A(0) = 1000$, we have $1000 = \left(\frac{25+C}{3}\right)^3$. It follows that $C = 5$, so

$$A(t) = \left(\frac{t + 50\,e^{-0.02\,t} + 10}{6}\right)^3.$$

20. a. The equation for the weight of the swordfish is a linear differential equation, so we first write

$$\frac{dw}{dt} = 0.015(1000 - w) = -0.015(w - 1000).$$

We make the substitution $z(t) = w(t) - 1000$, giving the differential equation $\frac{dz}{dt} = -0.015z$ with the initial condition $z(0) = w(0) - 1000 = -1000$. Thus, $z(t) = -1000\,e^{-0.015\,t}$. It follows that

$$w(t) = 1000 - 1000\,e^{-0.015\,t}.$$

The swordfish reaches 70 kg when $1000 - 1000\,e^{-0.015\,t} = 70$ or $e^{0.015\,t} = \frac{1000}{930}$. Thus, it takes $t = \frac{200}{3}\ln\left(\frac{100}{93}\right) \approx 4.838$ yrs to reach maturity.

b. The mercury (Hg) accumulates in swordfish according to the differential equation, which is a time varying equation. It follows that upon integration that

$$H(t) = 0.01 \int (1000 - 1000\,e^{-0.015\,t})dt = 10t + \frac{2000}{3}\,e^{-0.015\,t} + C.$$

With the initial condition $H(0) = 0$, the solution becomes

$$H(t) = 10t + \frac{2000}{3}\,e^{-0.015\,t} - \frac{2000}{3}.$$

From this equation, it follows that $H(3) = 0.665$ and $H(20) = 27.2$ mg of Hg.

c. The formula for the concentration of Hg, $c(t)$ (in μg/g) in swordfish satisfies

$$c(t) = H(t)/w(t) = \frac{10t + \frac{2000}{3}\,e^{-0.015\,t} - \frac{2000}{3}}{1000 - 1000\,e^{-0.015\,t}}.$$

It follows that $c(3) = 0.0151$ and $c(20) = 0.105$ μg/g.

21. a. Write the differential equation $\frac{dw}{dt} = -0.2(w - 80)$, then $z(t) = w(t) - 80$. It follows that

$$\frac{dz}{dt} = -0.2z, \qquad z(0) = -80,$$

with the solution $z(t) = -80e^{-0.2t} = w(t) - 80$. Thus,

$$w(t) = 80\left(1 - e^{-0.2t}\right).$$

For a 40 kg alligator, $w(t) = 40 = 80\left(1 - e^{-0.2t}\right)$ or $40 = 80e^{-0.2t}$, so $e^{0.2t} = 2$ or $0.2t = \ln(2)$. Thus, $t = 5\ln(2) \approx 3.47$ years. b. The pesticide accumulation is given by

$$\frac{dP}{dt} = 600\left(80\left(1 - e^{-0.2t}\right)\right), \qquad P(0) = 0.$$

The solution is given by

$$P(t) = 48,000\int\left(1 - e^{-0.2t}\right)dt = 48,000\left(t + 5e^{-0.2t}\right) + C.$$

The initial condition gives $P(0) = 0 = 240,000 + C$, so $C = -240,000$. Hence,

$$P(t) = 48,000\left(t + 5e^{-0.2t}\right) - 240,000.$$

The amount of pesticide in the alligator at age 5 is $P(5) = 48,000\left(5 + 5e^{-1}\right) - 240,000 = 240,000e^{-1} \approx 88291$ μg.

c. The pesticide concentration for a 5 year old alligator is

$$c(5) = \frac{P(5)}{1000w(5)} = \frac{88,291}{80,000\left(1 - e^{-1}\right)} \approx 1.75 \text{ ppm}.$$

22. a. The differential equation can be written:

$$\frac{dc}{dt} = -0.004(c - 15),$$

so we make the substitution $z(t) = c(t) - 15$. Since $c(0) = 0$, it follows that $z(0) = -15$. The solution of the substituted equation is given by:

$$\begin{aligned} z(t) &= -15e^{-0.004t} = c(t) - 15 \\ c(t) &= 15 - 15e^{-0.004t}. \end{aligned}$$

The limiting concentration satisfies:

$$\lim_{t\to\infty} c(t) = 15 \text{ mg/m}^3.$$

b. We begin by separating variables, which gives:

$$\begin{aligned} \int\frac{dc}{c - 15} &= -0.001\int(4 - \cos(0.0172t))\,dt \\ \ln(c(t) - 15) &= -0.001\left(4t - \frac{\sin(0.0172t)}{0.0172}\right) + C \\ c(t) &= 15 + Ae^{-0.001\left(4t - \frac{\sin(0.0172t)}{0.0172}\right)} \end{aligned}$$

It is easy to see that the initial condition $c(0) = 0$ implies that $A = -15$. Thus, the solution to this problem is given by:

$$c(t) = 15 - 15\, e^{-0.001(4t - 58.14\sin(0.0172t))}$$

23. a. We separate variables, so

$$\int M^{-3/4} dM = -k \int dt \quad \text{or} \quad 4M^{1/4} = -kt + 4C$$

$$M(t) = \left(C - \frac{k}{4}t\right)^4$$

From the initial condition, $M(0) = 16 = C^4$, it follows that $C = 2$. From the information that $M(10) = 1 = (2 - 10k/4)^4$, we have $k = 0.4$, so

$$M(t) = (2 - 0.1t)^4.$$

The fruit vanishes in 20 days.

b. We separate variables again to find:

$$\int M^{-3/4} dM = -0.8 \int e^{-0.02t} dt \quad \text{or} \quad 4M^{1/4} = \frac{0.8}{0.02} e^{-0.02t} + 4C$$

$$M(t) = \left(10e^{-0.02t} + C\right)^4.$$

From the initial condition, $M(0) = 16 = (10 + C)^4$, it follows that $C = -8$, so

$$M(t) = \left(10e^{-0.02t} - 8\right)^4.$$

Solving $10e^{-0.02t} = 8$, which is when the fruit vanishes, we find $t = 50\ln(5/4)$. Thus, the fruit vanishes in 11.157 days.

24. a. The general solution to the Malthusian growth problem with the initial condition $P(0) = 60$ is

$$P(t) = 60\, e^{rt}.$$

We are given that 2 weeks later $P(2) = 80 = 60\, e^{2r}$, so it follows that $r = \frac{1}{2}\ln\left(\frac{4}{3}\right) = 0.14384$. This gives the solution:

$$P(t) = 60\, e^{0.14384\, t}.$$

It is easy to see that the population doubles when $120 = 60\, e^{0.14384\, t}$, so $0.14384\, t_d = \ln(2)$ or the doubling time is

$$t_d = \frac{\ln(2)}{r} = 4.819 \text{ weeks.}$$

b. We begin by separating variables, so the general solution satisfies:

$$\int \frac{dP}{P} = \int (a - b\,t)\, dt \quad \text{or} \quad \ln(P(t)) = a\,t - \frac{bt^2}{2} + C \quad \text{or} \quad P(t) = e^C e^{a\,t - \frac{bt^2}{2}}.$$

Since the initial value is $P(0) = 60$, it follows that $e^C = 60$. Thus,

$$P(t) = 60\, e^{a\, t - \frac{bt^2}{2}}.$$

We now use the data at $t = 2$ and 4 weeks. It follows from the solution above that

$$\begin{aligned} 80 &= 60\, e^{2\, a - 2\, b} \\ 90 &= 60\, e^{4\, a - 8\, b}. \end{aligned}$$

We rearrange the terms and take logarithms of both sides to get

$$\begin{aligned} 2\, a - 2\, b &= \ln\left(\frac{4}{3}\right) \\ 4\, a - 8\, b &= \ln\left(\frac{3}{2}\right). \end{aligned}$$

We solve these equations simultaneously to obtain

$$2\, b = \ln\left(\frac{4}{3}\right) - \frac{1}{2}\ln\left(\frac{3}{2}\right),$$

so $b = 0.042475$. But $a = b + \frac{1}{2}\ln(4/3)$ or $a = 0.1863$. It follows that the solution is

$$P(t) = 60\, e^{0.1863\, t - 0.021237\, t^2}.$$

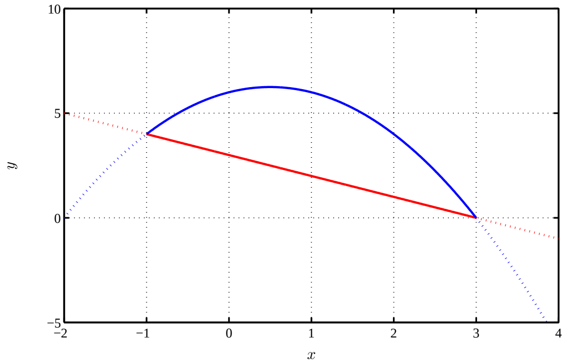
The population reaches a maximum when the derivative is zero, which occurs when $t_{max} = \frac{a}{b} = 4.3865$, so the maximum population is $P(t_{max}) = 90.286$.

25. a. From the equation of the line, $y = 3 - x$, the x and y-intercepts are easily seen to be $(3, 0)$ and $(0, 3)$, respectively. Also, the slope of the line is $m = -1$. The equation for the parabola is $y = 6 + x - x^2 = -(x + 2)(x - 3)$. From this it is easy to see that the x-intercepts are $(-2, 0)$ and $(3, 0)$, while the y-intercept is $(0, 6)$. The vertex of the parabola has its x-coordinate halfway between the x-intercepts, thus the vertex is $\left(\frac{1}{2}, 6\,\frac{1}{4}\right)$.

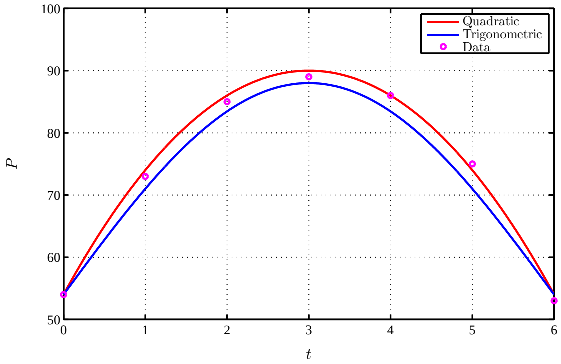
Setting the two equations equal to each other, $3-x = 6+x-x^2$ or $x^2-2x-3 = (x+1)(x-3) = 0$. We find that the points of intersection are $(-1, 4)$ and $(3, 0)$. The graph can be seen below. The area of interest is enclosed in the solid lines.

b. The area between the two curves is given by

$$\begin{aligned} \int_{-1}^{3} \left(6 + x - x^2 - (3 - x)\right) dx &= \int_{-1}^{3} 3 + 2\, x - x^2 dx \\ &= \left(3x + x^2 - \frac{x^3}{3}\right)\Bigg|_{x=-1}^{3} \\ &= 9 + 9 - 9 + 3 - 1 - \frac{1}{3} = \frac{32}{3} \end{aligned}$$



Problem 25



Problem 26

26. a. The maximum population for $P(t) = 54 + 24\,t - 4\,t^2$ is found by differentiating with $P'(t) = 24 - 8\,t$, which is zero at $t = 3$. This gives a maximum population of $P(3) = 90$.

b. The maximum population for $Q(t) = 54 + 34\sin\left(\frac{\pi}{6}t\right)$ occurs when $t = 3$, which is when the argument of the sine function is at $\pi/2$. This gives a maximum of $Q(3) = 88$. A graph of the function (along with the other model and the data) is above.

c. The average of the data is $\frac{54+73+85+89+86+75+53}{7} = 73.57$. The averages from the integrals are

$$\begin{aligned} P_{ave} = \frac{1}{6}\int_0^6 P(t)dt &= \frac{1}{6}\int_0^6 (54 + 24\,t - 4\,t^2)dt \\ &= \frac{1}{6}\left(54\,t + 12\,t^2 - \frac{4}{3}t^3\right)\Big|_{x=0}^{6} \\ &= \frac{1}{6}\left(54(6) + 72(6) - 48(6)\right) = 78 \end{aligned}$$

and

$$\begin{aligned} Q_{ave} = \frac{1}{6}\int_0^6 Q(t)dt &= \frac{1}{6}\int_0^6 (54 + 34\,\sin(1/6\,\pi\,t))dt \\ &= \frac{1}{6}\left(54\,t - \frac{204}{\pi}\,\cos(1/6\,\pi\,t)\right)\Big|_{x=0}^{6} \\ &= \frac{1}{6}\left(54(6) - \frac{204}{\pi}(\cos(\pi) - \cos(0))\right) = 54 + \frac{68}{\pi} = 75.65. \end{aligned}$$