Spring 2016 **Math 531 Take-Home Final** **Due Thurs, May 12, 2016 by 5 PM**

I, ________________________________(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

For all of the problems below, perform all integrations that can be readily be done. Use orthogonality to eliminate any zero coefficients. State clearly your reference for any short-cutted solutions to Sturm-Liouville problems.

1. A better model for the string problem is given by the nonhomogeneous partial differential equation:

$$u_{tt} + 2ku_t = c^2 u_{xx} - g, \qquad t > 0 \quad \text{and} \quad 0 < x < 1,$$

where k is a small positive constant ($k \ll c\pi$), which accounts for air resistance, and g is the acceleration due to gravity on the string. Assume that the ends of the string are fixed with $u(0,t) = 0$ and $u(1,t) = 0$.

a. Find the equilibrium position for the string.

b. Suppose that the initial displacement is the same as the equilibrium position and the initial velocity is 1 at each point of the string, *i.e.,* $u_t(x, 0) = 1$. Find $u(x,t)$ and determine the limit of $u(x,t)$ as $t \to \infty$.

2. Consider the nonhomogeneous partial differential equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-2t}\cos\left(\frac{2\pi x}{L}\right), \qquad t > 0 \quad \text{and} \quad 0 < x < L,$$

where $k(2\pi/L)^2 \neq 2$ and with boundary conditions:

$$\frac{\partial}{\partial x}u(0,t) = 0 \quad \text{and} \quad \frac{\partial}{\partial x}u(L,t) = 0.$$

Assume an initial condition:

$$u(x, 0) = f(x).$$

Use the method of eigenfunction expansion with $u(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x)$, where $\phi_n(x)$ are the appropriate eigenfunctions corresponding to the homogeneous boundary conditions above, to solve this problem.

3. Solve the initial value problem for the nonhomogeneous heat equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) + q_0 e^{-t} r\,\sin(5\theta), \qquad 0 < r < 1, \quad 0 < \theta < \frac{\pi}{2}, \quad t > 0,$$

with the boundary conditions:

$$u(1, \theta, t) = 0, \quad u(r, 0, t) = 0, \quad u_\theta(r, \pi/2, t) = 0,$$

and initial condition:

$$u(r, \theta, 0) = f_0 r^3 \sin(3\theta).$$

(Note: f_0 and q_0 are constants.)

4. a. Consider an infinite rod, which satisfies the partial differential equation given by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma(u - T_e), \qquad -\infty < x < \infty, \quad t > 0$$

with initial conditions:

$$u(x, 0) = \begin{cases} 0, & |x| < c \\ T_e, & |x| > c \end{cases}.$$

Briefly describe what this is happening physically to this infinite rod. Clearly explain each term and to what the initial conditions correspond.

b. Solve this problem for $u(x, t)$.

c. Use your solution to create a 3D plot of $u(x, t)$ with $t \in [0.001, 10]$ and $x \in [-10, 10]$ with the parameters $k = 1$, $T_e = 25$, $c = 2$, and $\gamma = 0.1$.

5. Find the solution for Laplace's equation in a semi-infinite strip;

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad x > 0, \quad 0 < y < b,$$

with the boundary conditions:

$$u(0, y) = T_0 \cos\left(\frac{2\pi y}{b}\right), \quad \frac{\partial}{\partial y}u(x, 0) = 0, \quad u(x, b) = \begin{cases} T_0, & 0 < x < a \\ 0, & x > a \end{cases}.$$