Spring 2010 Math 121a Name Key

Give your answers to **at least 4 significant figures** whenever possible.

1. (20pts) Differentiate the following functions (you **don't** have to simplify):

a. $f(t) = 2(t^2 - e^{-2t})^3 + 3\ln(t^2+4) - \frac{4}{\sqrt{t}}$.

4, 4, 2 $f'(t) = 6(t^2 - e^{-2t})^2(2t + 2e^{-2t}) + \frac{6t}{t^2+4} + 2t^{-3/2}$

b. $g(x) = \frac{e^{2x}}{x^4 + x^2 - 5} + (x^2+3)e^{-x^2} - 2x$.

4, 5, 1 $g'(x) = \frac{(x^4+x^2-5)2e^{2x} - e^{2x}(4x^3+2x)}{(x^4+x^2-5)^2} + (x^2+3)e^{-x^2}(-2x) + 2xe^{-x^2} - 2$

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2. (10pts) **a.** A frog needs to jump over a garden wall to reach a mate. The wall is 20 cm high. Assume that the frog jumps the wall with just enough vertical velocity, v_0 to clear it. If the height (in cm) of the frog is given by

$$h(t) = v_0 t - 490t^2,$$

then determine the velocity of the frog at any time, $v(t) = dh/dt$, where the velocity depends on v_0 and t.

3 $v(t) = v_0 - 980t$ cm/sec

b. Use the height of the garden wall and the function describing the height of the frog, $h(t)$, to determine the minimum vertical velocity, v_0, needed to clear the wall. Also, determine how long the frog is in the air, assuming it lands in the garden (height $h = 0$) on the other side of the wall.

$v(t) = 0$ (max) $\Rightarrow t_{max} = \frac{v_0}{980}$ $h(t_{max}) = \frac{v_0^2}{980} - 490\left(\frac{v_0}{980}\right)^2 = \frac{v_0^2}{1960} = 20$

$v_0 = \sqrt{20(1960)}$

5, 2 $v_0 =$ 197.99 cm/sec Time in air $=$ 0.4041 sec

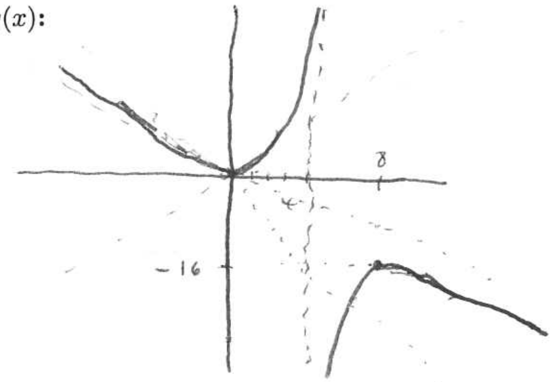
Time in air $= 2 \cdot t_{max} = \frac{2 \cdot 197.99}{980}$

3. (32pts) Sketch the graph of the following functions. Give the x and y-intercepts, and any asymptotes. Find the derivative and its critical point(s) (including the x and y values). Indicate whether it is a local maxima or minima. For function. (If the function does not have a particular asymptote, extrema or x or y-intercept, indicate "NONE").

a. $y = \dfrac{x^2}{4-x}$.

Graph of $y(x)$:

2



1 x-intercept(s) 0

1 y-intercept 0

2 Vertical asymptote(s) $x = 4$

1 Horizontal asymptote(s) None

3 $y'(x) = \dfrac{(4-x)(2x)+x^2}{(4-x)^2} = \dfrac{8x - x^2}{(4-x)^2}$

$x_c = 0, 8$

$y(8) = \dfrac{64}{-4} = -16$

2, 1 $x_{max} =$ 8 $\quad y(x_{max}) =$ -16

2, 1 $x_{min} =$ 0 $\quad y(x_{min}) =$ 0

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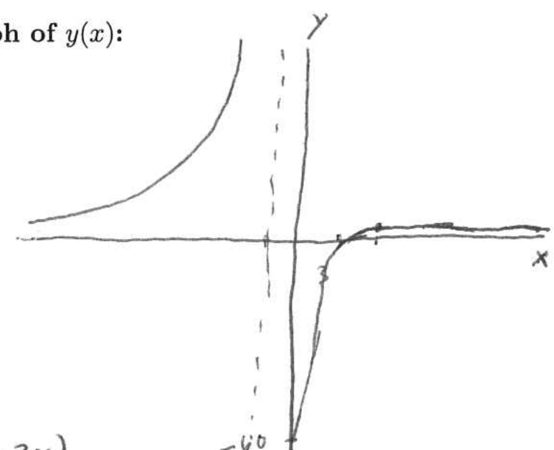
b. $y = \dfrac{20(x-3)}{(1+x)^3}$.

Graph of $y(x)$:

2



1 x-intercept(s) 3

1 y-intercept is -60

1 Vertical Asymptote(s) $x = -1$

2 Horizontal Asymptote(s) $y = 0$

3 $y'(x) = 20\dfrac{(1+x)^3 - (x-3)(3)(1+x)^2}{(1+x)^6} = 20\dfrac{(10-2x)}{(1+x)^4}$

3, 1 $x_{max} =$ 5 $\quad y(x_{max}) =$ 0.1852

$x_c = 5$

1, 1 $x_{min} =$ None $\quad y(x_{min}) =$ None

$y(5) = \dfrac{40}{6^3} = 0.1852$

4. (25pts) *The ASPCA Complete Dog Care Manual* (p. 55) states that the normal food requirement for a 5 kg dog is 450 calories. It gives the normal requirement for a 30 kg dog as 1700 calories. (Use **4 significant figures** for all calculations.)

a. Assume linear relationship between the weight of the dog (in kg) (W) and the normal food requirement (in calories) (F) that it consumes

$$F = mW + b.$$

Use the data above to find the constants m and b in the model above.

$m = \dfrac{1700 - 450}{30 - 5} = \dfrac{1250}{25} = 50$

3 $m =$ 50

3 $b =$ 200

$b = 450 - 5(50) = 200$

b. Assume there is a power law relationship between the weight of the dog (W) and the normal food requirement (F) that it consumes

$$F = kW^a. \qquad \ln F = \ln(k) + a\ln(W)$$

Use the data above to find the constants a and k in the power law or allometric model above.

W	ln(W)	F	ln(F)
5	1.60944	450	6.10925
30	3.40120	1700	7.43838

2 $a = \dfrac{\ln(F_2) - \ln(F_1)}{\ln(W_2) - \ln(W_1)} = \dfrac{7.43838 - 6.10925}{3.40120 - 1.60944} = 0.7418$

$\ln(k) = \ln(F) - a\ln(W) = 6.10925 - 0.7418(1.60944) = 4.91536$

3 $k =$ 136.37

$k = e^{4.91536} = 136.37$

3 $a =$ 0.7418

c. Use both models (linear and allometric) to find the amount of feed consumed by a 20 kg dog. Also, estimate the weight of a dog that normally consumes 1000 calories of food using both models. Which model gives the better predictions and why?

For **Linear Model**:

$50(20) + 200 = 1200$

2 If $W = 20$ kg, then $F =$ 1200 calories,

$1000 = 50W + 200 \qquad W = \dfrac{800}{50}$

3 If $F = 1000$ calories, then $W =$ 16 kg.

For **Allometric Model**:

$136.37(20)^{0.7418}$

2 If $W = 20$ kg, then $F =$ 1258 calories,

$1000 = 136.37\, W^{0.7418}$

3 If $F = 1000$ calories, then $W =$ 14.671 kg.

$W = \left(\dfrac{1000}{136.37}\right)^{1/0.7418}$

d. Which model provides the better estimate?

1 **Circle one:** **Linear Model** **Allometric Model**

5. (20pts) **a.** The population of Poland in 1950 was about 24.82 million, while in 1970, it was about 32.53 million. Assume that the population is growing according to the discrete Malthusian growth equation

$$P_{n+1} = (1+r)P_n, \quad \text{with} \quad P_0 = 24.82,$$

where P_0 is the population in 1950 and n is in decades. Use the population in 1970 (P_2) to find the value of r (to **4 significant figures**). Find how long it would take for this population to double.

$24.82(1+r)^2 = 32.53$ $\qquad$ $1+r = \left(\frac{32.53}{24.82}\right)^{1/2} = 1.14483$

$t_d = \frac{\ln(2)}{\ln(1.14483)} = 5.12$ decades

2,2 $\quad r =$ 0.14483 $\quad$ Doubling time = 51.2 (in years)

b. Estimate the population in 2000 based on the Malthusian growth model. Given that the population in 2000 was 38.65 million, find the percent error between the actual and predicted values.

$24.82(1.14483)^5$ $\qquad$ $100 \cdot \frac{(48.81 - 38.65)}{38.65}$

2,2 $\quad$ Population in 2000 = 48.81 $\quad$ and $\quad$ Percent Error = 26.29 %

c. A better model fitting the census data for Poland is a logistic growth model given by

$$P_{n+1} = F(P_n) = 1.45P_n - 0.0112P_n^2,$$

where again n is in decades after 1950. Estimate the populations in 1960 and 1970 by computing P_1 and P_2, where $P_0 = 24.82$.

1,1 $\quad P_1 =$ 29.09 $\quad$ and $\quad P_2 =$ 32.70

d. Find the equilibrium for this logistic growth model. Calculate the derivative of $F(P)$ and evaluate it at the larger of the equilibria. What does this value say about the behavior of the solution near this equilibrium?

$P_e = 1.45P_e - 0.0112P_e^2 \Rightarrow P_e = 0$

$P_e = \frac{0.45}{0.0112}$

1,2 $\quad P_{1e} =$ 0 $\quad$ and $\quad P_{2e} =$ 40.18 $\quad (P_{1e} < P_{2e})$

3,2 $\quad F'(P) =$ 1.45 − 0.0224 P $\quad F'(P_{2e}) =$ 0.55

1,1 $\quad$ **Stable** or **Unstable** $\qquad$ **Monotonic** or **Oscillatory**

6. (20pts) **a.** A woman with a chronic lung problem breathes a supply of air enriched with helium (400 ppm). The initial concentration of helium in her lungs is $c_0 = 400$, and the measurement of helium in her lungs after her first breath is $c_1 = 352$. If the concentration of helium in the room is negligible, then an appropriate model for the concentration of helium (He) is given by the model:

$$c_{n+1} = (1-q)c_n,$$

where c_n is the concentration of He in the n^{th} breath and q is the fraction of air exchanged. Use the data for c_0 and c_1 to estimate the value of q. Then use this model to estimate the concentration of He in the 4^{th} breath (c_4). Determine how many breaths it takes for the He concentration to fall to one half (200 ppm) the original concentration.

$352 = (1-q)400 \qquad 1-q = \frac{352}{400} = 0.88 \qquad q = 0.12$

$c_n = c_0(1-q)^n$

2, 2 $\quad q =$ 0.12 $\qquad c_4 =$ 239.88

$= 400(0.88)^n$

2 $\quad$ Concentration He = 200 ppm when $n =$ 5.422

$200 = 400(0.88)^n$

$n = \ln(1/2)/\ln(0.88)$

b. It is determined that there is Helium in the room. The concentration of Helium in the room, γ, is not known, but assumed to be constant. Below is a table of the patient's first two breaths after resuming normal breathing in the room.

Breath Number	0	1	2
Conc. of He (ppm)	400	352	310

Assume a breathing model of the form:

$$c_{n+1} = (1-q)c_n + q\gamma.$$

Use the data above to find the constants, q and γ, the ambient concentration of Helium. Then determine the concentration of Helium in the next two breaths, c_3 and c_4. Assuming that this is the better model, find the percent error between the model in Part a and this model for the estimate of c_4.

$352 = 400(1-q) + q\gamma$

$310 = 352(1-q) + q\gamma$

$48(1-q) = 42 \qquad 1-q = 7/8 \qquad q = 1/8$

$352 = 400(7/8) + q\gamma \qquad q\gamma = 2 \qquad \gamma = 16$

3, 3 $\quad q =$ 0.125 $\qquad \gamma =$ 16

2, 1, 2 $\quad c_3 =$ 273.25 and $c_4 =$ 241.09 $\qquad$ % Error at $c_4 =$ −0.503%

$100\,\frac{(239.88 - 241.09)}{241.09}$

c. Find the equilibrium concentration of Helium in the subject's lungs based on the breathing model in Part b. What is the stability of this equilibrium concentration?

$c_e = (1-q)c_e + q\gamma \Rightarrow c_e = \gamma = 16$

2, 1 $\quad$ Equilibrium $c_e =$ 16 $\qquad$ **STABLE** or **UNSTABLE** (Circle one)

7. (23pts) Body temperatures of animals undergo circadian rhythms. A subject's temperature is measured from 8 AM until midnight, and his body temperature, T (in °F), is best approximated by the cubic polynomial

$$T(t) = 0.002(20000 - t^3 + 45\,t^2 - 600\,t),$$

where t is in hours.

a. Find the rate of change in body temperature $T'(t)$. What is the rate of change in body temperature at noon $t = 12$? Also, compute $T''(t)$. When is the rate of change in body temperature per hour increasing the most and what is that maximum rate of increase?

$T'(t) = 0.002\,(-3t^2 + 90t - 600) = -0.006\,(t^2 - 30t + 200) =$
$-0.006\,(t-10)(t-20)$

3, 2 $\quad T'(t) = \underline{-0.006(t^2 - 30t + 200)}$ $\quad T'(12) = \underline{0.096}$

3 $\quad T''(t) = \underline{-0.012\,(t - 15)}$

2, 1 $\quad$ Rate of maximum increase at $t_{inc} = \underline{15}$ $\quad T'(t_{inc}) = \underline{0.15}$

b. Use the derivative to find when the maximum temperature of the subject occurs and when the minimum temperature of the subject occurs. What are the body temperatures at those times?

3, 1 $\quad t_{max} = \underline{20}$ $\quad T(t_{max}) = \underline{36.0}$

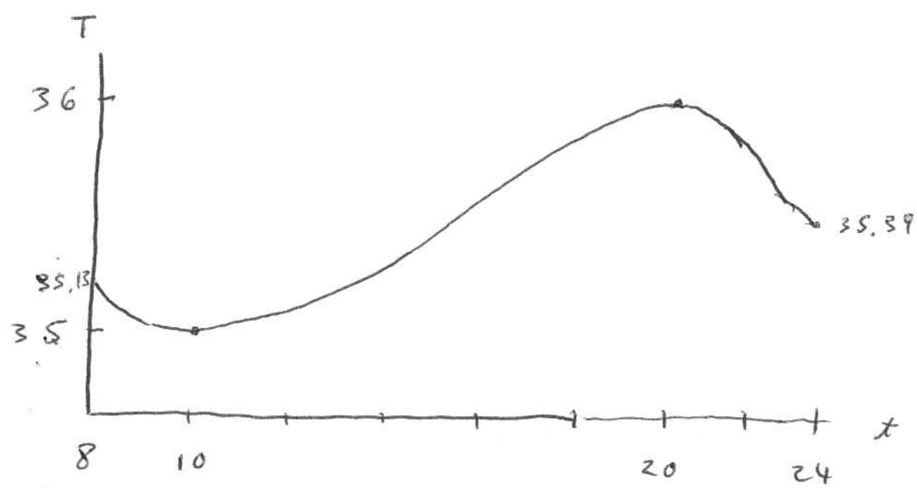
3, 1 $\quad t_{min} = \underline{10}$ $\quad T(t_{min}) = \underline{35.0}$

c. Sketch a graph of this polynomial fit to the body temperature. Show clearly the maximum and minimum body temperature on your graph and include the body temperature at the beginning of the study ($t = 8$) and at the end ($t = 24$).

1, 1 $\quad T(8) = \underline{35.136}$ $\quad T(24) = \underline{35.392}$

Graph of $T(t)$:

2



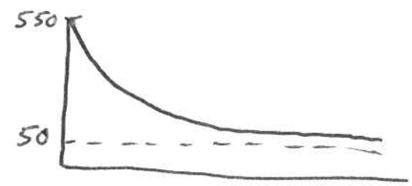
8. (23pts) **a.** It has been shown that iron is the primary limiting nutrient in open ocean waters. There are currently a number of experiments to see if seeding the ocean with iron can create an algal bloom that fixes CO_2 (to remove this greenhouse gas). Soluble iron that is dumped into the ocean is rapidly used by algae, which are consumed by other organisms. Suppose that at $t = 0$, a research vessel from Scripps Institute of Oceanography dumps a large amount of soluble iron. Measurements from a trailing ship indicate that the concentration of iron remaining in the water (not in the algae) satisfies the equation:

$$F(t) = 500\,e^{-0.23t} + 50 \text{ ppm},$$

where t is in days. Find how long it takes for the amount of soluble iron to reach the level of 100 ppm remaining. Sketch a graph of F showing the F-intercept and the horizontal asymptote.

2 $F(t) = 100$ when $t =$ 10.011

2 **Graph of $F(t)$:**



1 $F(0) =$ 550

2 Horizontal Asymptote at $F =$ 50

$100 = 500e^{-0.23t} + 50 \Rightarrow e^{0.23t} = 10 \Rightarrow t = \dfrac{\ln(10)}{0.23}$

b. Find the derivative $\dfrac{dF}{dt}$. Determine the rate of change of soluble iron at $t = 2$.

2, 1 $F'(t) =$ $-115\,e^{-0.23t}$ $F'(2) =$ -72.598

c. As noted above the algae rapidly blooms, then fades as the iron passes to organisms higher in the food web. Suppose that samples of the sea water give a population of algae, $P(t)$, (in thousands/cc) satisfying the following equation:

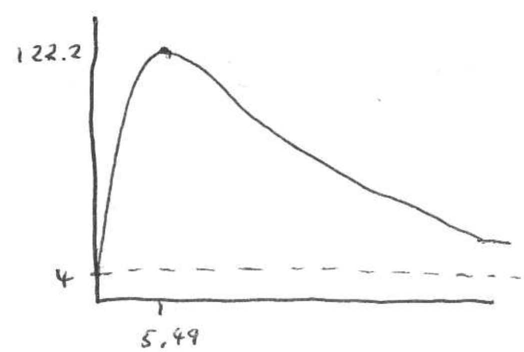
$$P(t) = 4 + 160(e^{-0.04t} - e^{-0.5t}),$$

where t is in days. Find the derivative $\dfrac{dP}{dt}$. Find when the algal population achieves its maximum concentration and determine what its maximum concentration is. Sketch a graph of P showing the P-intercept, the maximum, and any horizontal asymptotes.

3 $P'(t) =$ $160(-0.04e^{-0.04t} + 0.5e^{-0.5t})$ $= -6.4e^{-0.04t} + 80e^{-0.5t}$

Graph of $P(t)$:

3, 2 $t_{max} =$ 5.491 $P(t_{max}) =$ 122.17



1 $P(0) =$ 4

2 Horizontal Asymptote at $P =$ 4

$0.04e^{-0.04t} = 0.5e^{-0.5t}$

$e^{0.46t} = \dfrac{50}{4}$

2

9. (27pts) Ricker's model is often used to study the population of fish. Let P_n be the population (in thousands) of a species of fish in years n and suppose that Ricker's model is given by

$$P_{n+1} = R(P_n) = 6.2P_n e^{-0.005\,P_n}.$$

a. Assume that the initial population is $P_0 = 25$, then determine the population of fish for the next two years (P_1 and P_2).

1, 1 $P_1 =$ 136.79 $P_2 =$ 427.96

b. Find $R'(P)$, then determine the maximum of this function (both P and $R(P)$ values). Sketch a graph of $R(P)$ with the identity function for $P \geq 0$, showing the intercepts and any horizontal asymptotes.

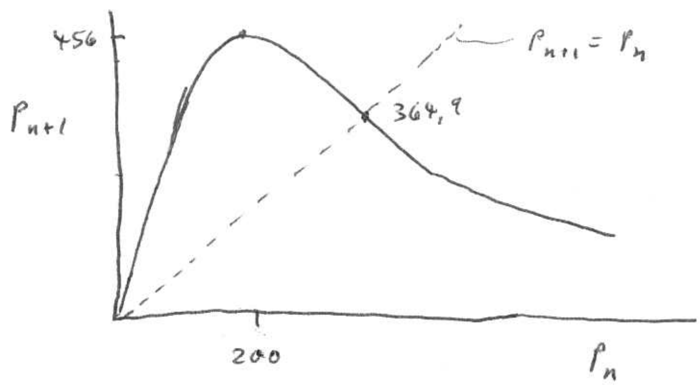
3 $R'(P) = 6.2\,e^{-0.005P}(1 - 0.005P)$

1, 1, 2 P-intercept 0 R-intercept 0 Horizontal Asymptote $R =$ 0

3, 1 $P_{max} =$ 200 $R(P_{max}) =$ 456.17

GRAPH:

2



c. Find all equilibria for Ricker's model and determine the stability of the equilibria. Justify your stability argument by evaluating the derivative of the updating function.

$P_e = 6.2P_e\,e^{-0.005P_e} \Rightarrow P_e = 0$ or $1 = 6.2\,e^{-0.005P_e}$

$e^{0.005P_e} = 6.2$

$P_e = 200\ln(6.2)$

1, 2 $P_{1e} =$ 0 $R'(P_{1e}) =$ 6.2

1, 1 **Stable** or **Unstable** **Monotonic** or **Oscillatory**

3, 2 $P_{2e} =$ 364.91 $R'(P_{2e}) =$ −0.8245

1, 1 **Stable** or **Unstable** **Monotonic** or **Oscillatory**