Calculus for the Life Sciences I

Lecture Notes – Velocity and Tangent

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Outline

1. Cats and Gravity
   - Falling Cats
   - Flight of a Ball
   - Salmon Ladder

2. Secant and Tangent Lines
   - Geometric view of Derivative
   - Velocity of Cat

Cats and Gravity

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Controversy as to whether Newton or Gottfried Leibnitz was the first to invent Calculus

Falling Cat

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- Cats have a very flexible spine for hunting
- This flexibility allows them to rotate rapidly during a fall

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  - From intermediate heights, the cat basically achieves terminal velocity, but the tension causes increased likelihood of severe or fatal injuries

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- **Velocity is the derivative of position**
- **Acceleration is the derivative of velocity**

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How long does this cat fall?

What is its velocity when it hits the ground?

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We will show that the cat has a velocity,

$$v(1) = -32 \text{ ft/sec} \quad (\text{about } 21.8 \text{ mph})$$

Flight of a Ball 1

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- The acceleration of gravity is $g = 980$ cm/sec^2

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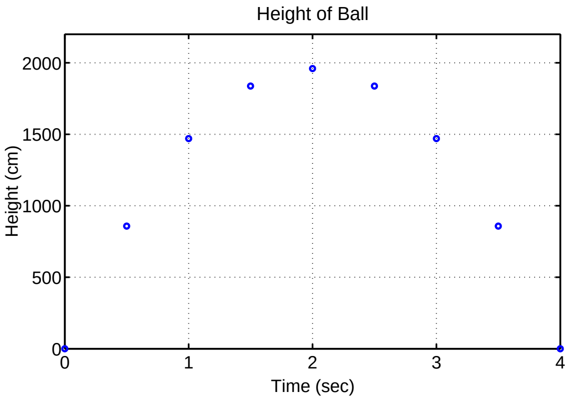
- It is thrown vertically with an initial velocity, $v(0) = 1960$ cm/sec
- The acceleration of gravity is $g = 980$ cm/sec^2
- The height of the ball for any time t $(0 \leq t \leq 4)$ is given by

$$h(t) = 1960\,t - 490\,t^2 = 0$$

Flight of a Ball

Graph of the height of a ball for $0 \leq t \leq 4$, showing position every 0.5 sec



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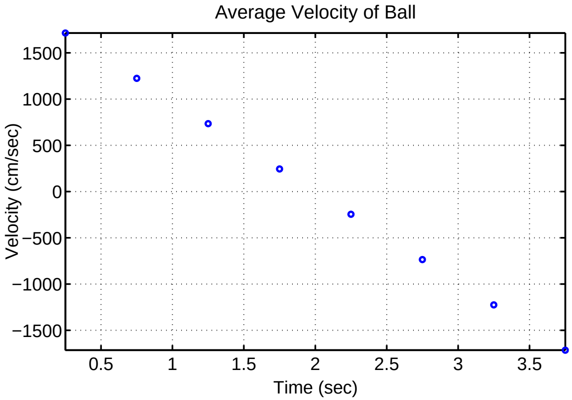
- The average velocity is the difference between the heights at two times divided by the length of the time period
- Associate the average velocity with the midpoint between each time interval

Height (t_1) $h(t_1)$	Height (t_2) $h(t_2)$	Average Time $t_a = (t_1 + t_2)/2$	Average Velocity $v(t_a) = \dfrac{h(t_2) - h(t_1)}{(t_2 - t_1)}$
$h(0) = 0$	$h(0.5) = 857.5$	$t_a = 0.5/2 = 0.25$	$v(0.25) = 1715$
$h(1.5) = 1837.5$	$h(2) = 1960$	$t_a = 1.75$	$v(1.75) = 245$
$h(3) = 1470$	$h(3.5) = 857.5$	$t_a = 3.25$	$v(3.25) = -1225$

Flight of a Ball

Graph of the velocity of a ball for $0 \leq t \leq 4$, showing velocity every 0.5 sec



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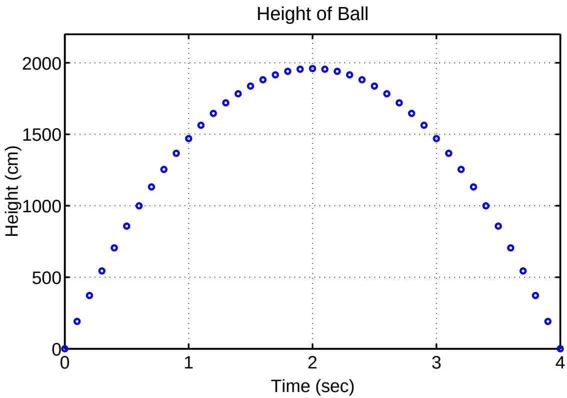
The **vertex of the parabola (maximum height of the ball)** is where the **velocity is zero (*t*-intercept)**

Flight of a Ball

Graph of the height of a ball for $0 \leq t \leq 4$, showing position every 0.1 sec



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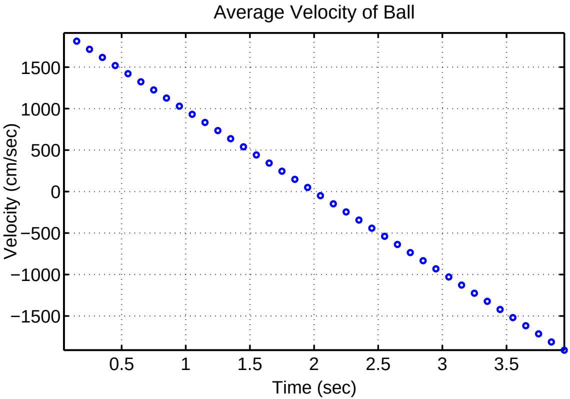
  - The distance between successive heights is now closer
  - But then the intervening time interval is also closer together
- The average velocity between $t_1 = 0.2$ and $t_2 = 0.3$ has $h(t_1) = 372.4$ cm and $h(t_2) = 543.9$ cm, so $v(0.25) = 1715$ cm/sec, the same as before

Flight of Ball

Graph of the velocity of a ball for $0 \leq t \leq 4$, showing velocity every 0.1 sec



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- This is specific to the **quadratic nature of the height function**
- Soon we will learn to take derivatives of more functions

Example – Flight of a Ball

A ball, which is thrown vertically with an initial velocity of 80 ft/sec and only the acceleration of gravity acting on the ball, satisfies the equation:

$$h(t) = 80\,t - 16\,t^2$$

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- Find the average velocity of the ball between $t = 0$ and $t = 1$ and associate this velocity with $t = 0.5$
- Repeat this process for each second of the flight of the ball, then sketch a graph of the average velocity as a function of time, t

Example – Flight of a Ball

Graph of Flight of a Ball

- We write the equation

$$h(t) = -16\,t(t-5)$$

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Example – Flight of a Ball 2

Graph of Flight of a Ball

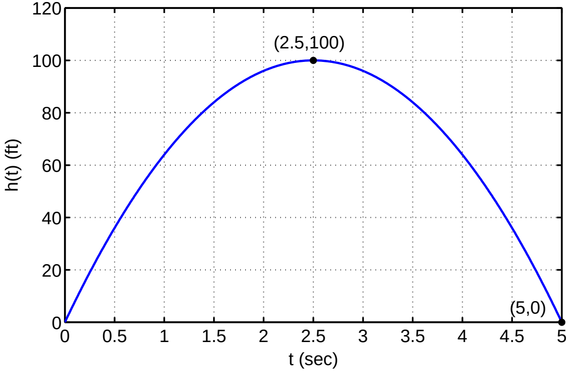
- We write the equation

$$h(t) = -16\,t(t-5)$$

- $h(t)$ is a parabola with t-intercepts at $t = 0$ and $t = 5$
- The vertex or maximum height of the ball occurs at the midpoint between these intercepts or $t = 2.5$ with $h(2.5) = 100$ ft

Example – Flight of a Ball

Graph of $h(t) = 80\,t - 16\,t^2$



Example – Flight of a Ball

Average velocity for the Flight of a Ball

- The average velocity for the ball between $t = 0$ and $t = 1$ is given by

$$v_{ave}(0.5) = \frac{h(1) - h(0)}{1 - 0} = \frac{64 - 0}{1} = 64 \text{ ft/sec}$$

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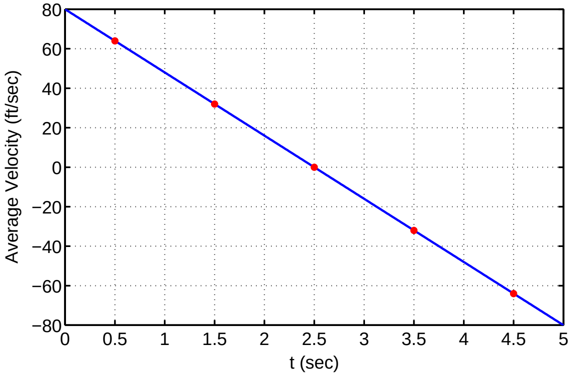
-

t	0.5	1.5	2.5	3.5	4.5
v_{ave}	64	32	0	-32	-64

Example – Flight of a Ball

The average velocities fall on the straight line

$$v_{ave}(t) = 80 - 32\,t$$



Example – Leaping Salmon 1

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Example – Leaping Salmon 1

- A river is dammed, and a salmon ladder is built to enable the salmon to bypass the dam and continue to travel upstream to spawn
- The vertical walls on the salmon ladder are 6 feet high
- The salmon has to leap vertically upwards over the wall
- The height of the salmon during its leap is given by

$$h(t) = v_0 t - 16\, t^2$$

Skip Example

Example – Leaping Salmon 2

- Let $v_0 = 20$ ft/sec. Sketch a graph of the height of the salmon $h(t)$, with time, showing clearly the maximum height and when the salmon can clear the wall

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- Repeat this process for each half-second of the leaping salmon, then sketch a graph of the average velocity as a function of time, t

Example – Leaping Salmon 2

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- Repeat this process for each half-second of the leaping salmon, then sketch a graph of the average velocity as a function of time, t
- Determine the minimum speed, v_0, that the salmon needs on exiting the water to climb the salmon ladder

Example – Leaping Salmon

Solution: The function $h(t)$ is a parabola,

$$h(t) = 20\,t - 16\,t^2 = 4\,t(5 - 4\,t)$$

Example – Leaping Salmon 3

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$$h(t) = 20\,t - 16\,t^2 = 4\,t(5 - 4\,t)$$

- The t-intercepts are $t = 0$ and $t = 1.25$
- The vertex occurs at $(0.625,\ 6.25)$

Example – Leaping Salmon 3

Solution: The function $h(t)$ is a parabola,

$$h(t) = 20\,t - 16\,t^2 = 4\,t(5 - 4\,t)$$

- The t-intercepts are $t = 0$ and $t = 1.25$
- The vertex occurs at $(0.625,\ 6.25)$
- The salmon can clear the wall when $h(t) = 6$, so

$$20\,t - 16\,t^2 = 6 \quad \text{or} \quad 8\,t^2 - 10\,t + 3 = 0$$

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$$20\,t - 16\,t^2 = 6 \quad \text{or} \quad 8\,t^2 - 10\,t + 3 = 0$$

- This can be factored to give

$$(2\,t - 1)(4\,t - 3) = 0$$

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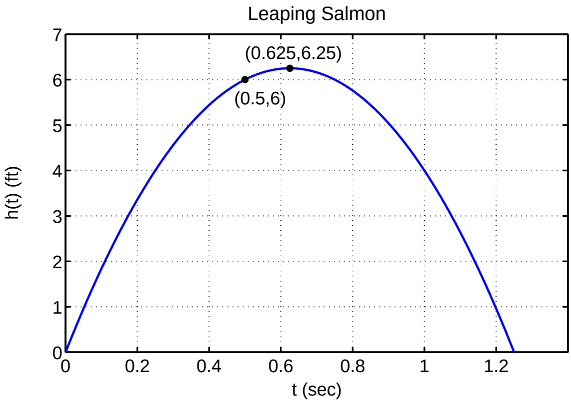
- The salmon can clear the wall at any time $\frac{1}{2} < t < \frac{3}{4}$ sec

Example – Leaping Salmon

Graph of $h(t) = 20\,t - 16\,t^2$



Example – Leaping Salmon 5

Solution (cont):

- The average velocity of the salmon between $t = 0$ and $t = 0.5$ is given by,

$$v(0.25) = \frac{h(0.5) - h(0)}{0.5} = \frac{(20(0.5) - 16(0.5)^2) - 0}{0.5} = 12 \text{ ft/sec}$$

Example – Leaping Salmon

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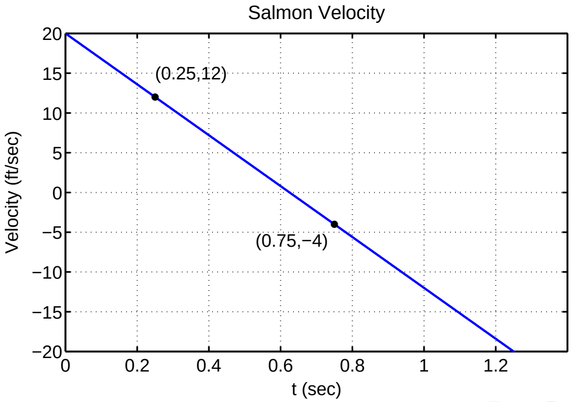
- The average velocity of the salmon between $t = 0.5$ and $t = 1$ is given by

$$v(0.75) = \frac{h(1) - h(0.5)}{0.5} = \frac{4 - 6}{0.5} = -4 \text{ ft/sec}$$

Example – Leaping Salmon

Graph of average velocity of the salmon satisfying

$$v_{ave}(t) = 20 - 32\,t$$



Example – Leaping Salmon

Solution (cont): The minimum speed, v_0, that the salmon needs to climb the fish ladder is the one that produces a maximum height of 6 ft

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$$t = \frac{-v_0}{2(-16)} = \frac{v_0}{32}$$
- Since we want the vertex to be 6 ft,
$$h\left(\frac{v_0}{32}\right) = v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 = \frac{v_0^2}{64} = 6.$$

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Solution (cont): The minimum speed, v_0, that the salmon needs to climb the fish ladder is the one that produces a maximum height of 6 ft

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$$h\left(\frac{v_0}{32}\right) = v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 = \frac{v_0^2}{64} = 6.$$
- $$v_0 = 8\sqrt{6} \approx 19.6 \text{ ft/sec}$$

Secant Lines and Tangent Line

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- The **tangent line** represents the best linear approximation to the curve near a given point
- Its slope is the derivative of the function at that point

Secant Lines and Tangent Line

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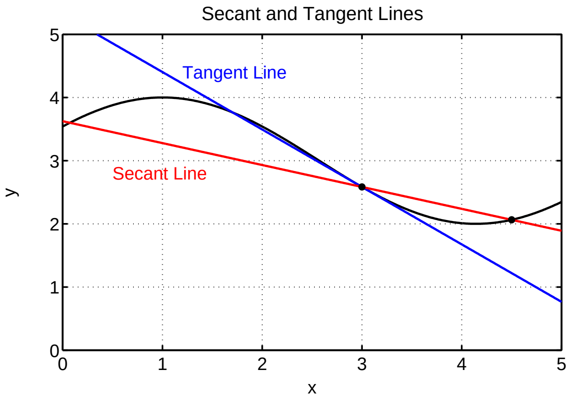
Definition: A **tangent line** for a curve is a line that touches the curve at exactly one point and provides the best approximation to the curve at that point.

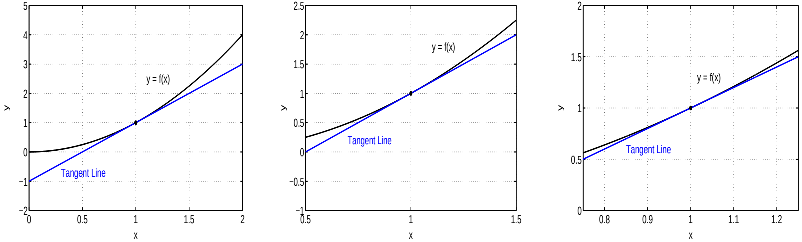
Secant Lines and Tangent Line

Graph showing Secant and Tangent Lines



Tangent Line

A **tangent line** represents the best linear approximation to the curve near a given point



Example – $y = x^2$

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- Find the equation of the **tangent line** at the point (1,1) on the graph

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- Find the equation of the **tangent line** at the point (1,1) on the graph
- A **secant line** is found by taking two points on the curve and finding the equation of the line through those points
- Create a **sequence of secant lines that converge to the tangent line** by taking the two points closer and closer together

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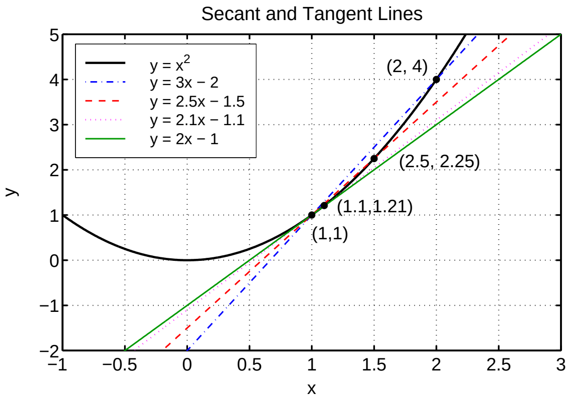
Example – $y = x^2$

Graph of $y = x^2$ with secant lines



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- The slope of the secant line through this point and the point (1,1) is

$$m = \frac{(1+2h+h^2) - 1}{(1+h) - 1} = \frac{2h + h^2}{h} = 2 + h$$

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- The formula for this secant line is

$$y = (2+h)x - (1+h)$$

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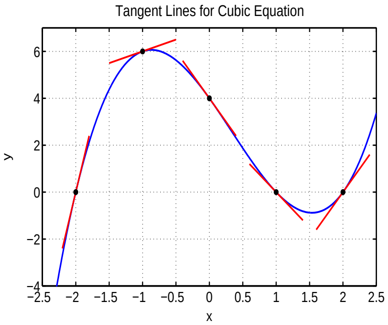
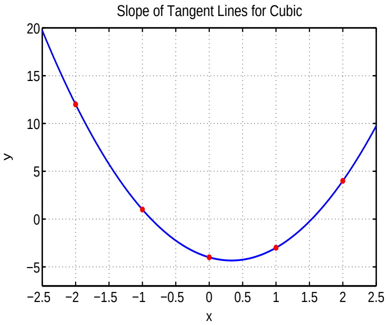
- Its not hard to see that the tangent line for $y = x^2$ at (1,1) is
$$y = 2x - 1$$
- The **slope of the tangent line is $m = 2$**
- The value of the **derivative of $y = x^2$ at $x = 1$**

Geometric view of Derivative

The geometric view of the tangent line is very easy to visualize

The graph on the left is $f(x)$ with tangent lines shown, while the graph on the right is the derivative of $f(x)$





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- The graph on the left is a cubic function, while the graph of its derivative is a quadratic
- As you approach a maximum (or minimum) for the cubic function, the value of the derivative goes to zero and the sign of the derivative function changes
- This is an important application of the derivative

Example – Secant Lines

Consider the function

$$f(x) = x^2 - x$$

Skip Example

Example – Secant Lines 1

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- Let all secant lines have the point, $x = 1$. Other points of the sequence have $x = 2$, $x = 1.5$, $x = 1.2$, $x = 1.1$, and $x = 1.01$

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- Find the derivative of $f(x)$ at $x = 1$ by finding the slope of the tangent line at $x = 1$
- Graph $f(x)$, the tangent line, and the secant lines

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Solution: This example examines secant lines for

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For $x = 1.5$, two points on the secant line are $(1, 0)$ and $(1.5, 0.75)$, which gives the secant line

$$y = 1.5x - 1.5$$

Example – Secant Lines

Solution (cont): Continuing the process:

When $x = 1.2$, two points on the secant line are $(1, 0)$ and $(1.2, 0.24)$, which gives the secant line

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For $x = 1.01$, two points on the secant line are $(1, 0)$ and $(1.01, 0.101)$, which gives the secant line

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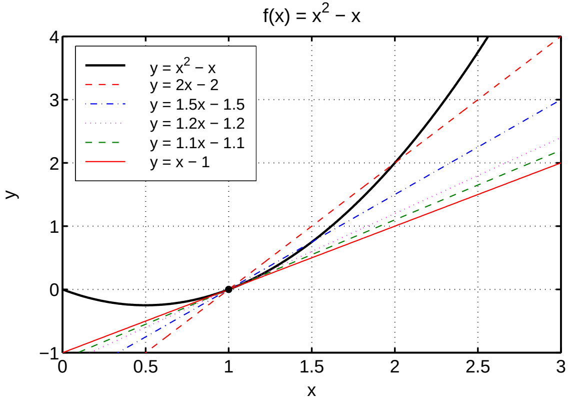
Example – Secant Lines 4

Solution (cont): The pattern in the sequence easily gives the tangent line

$$y = x - 1$$



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Since patterns cannot always be recognizable, we need a better way to compute the derivative

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Solution (cont): Let's find the slope of the secant line through the points

$$(1, f(1)) = (1, 0) \quad \text{and} \quad (1+h, f(1+h))$$

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Since $f(1+h) = (1+h)^2 - (1+h) = h^2 + h$, the slope of the secant line is

$$m(h) = \frac{(h^2+h) - 0}{(1+h) - 1} = \frac{h^2+h}{h} = 1 + h$$

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As $h \to 0$, $m(h) \to 1$

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As $h \to 0$, $m(h) \to 1$

It follows that the slope of the tangent line is **1**, which is the derivative of $f(x)$ at $x = 1$

Velocity of Falling Cat

Return to the cat falling from a 16 ft tree limb, where

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Since $h(1+z) = 16 - 16(1+z)^2 = -32\,z - 16\,z^2$

$$v_{ave} = \frac{h(1+z) - h(1)}{(1+z) - 1} = \frac{-32z - 16z^2}{z}$$

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As $z \to 0$, $v_{ave} \to -32$, so the cat hits the ground at a velocity of -32 ft/sec ($\simeq 21.8$ mph)

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- Let h get small and determine the slope of the tangent line through $(2, 2)$, which gives the value of the derivative of $f(x)$ at $x = 2$

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Solution: The slope of the secant line is

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$$= \frac{\sqrt{2+h+2} - \sqrt{2+2}}{h} = \frac{\sqrt{4+h} - 2}{h}$$

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Solution: The slope of the secant line is

$$\begin{aligned} m(h) &= \frac{f(2+h)-f(2)}{(2+h)-2} \\ &= \frac{\sqrt{2+h+2}-\sqrt{2+2}}{h} = \frac{\sqrt{4+h}-2}{h} \\ &= \left(\frac{\sqrt{4+h}-2}{h}\right)\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right) \end{aligned}$$

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&= \left(\frac{\sqrt{4+h}-2}{h}\right)\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right) \\
&= \frac{4+h-4}{h(\sqrt{4+h}+2)} \\
&= \frac{1}{\sqrt{4+h}+2}
\end{aligned}$$

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In the formula above, as $h \to 0$, the slope of secant line, m, approaches

$$m_t = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

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In the formula above, as $h \to 0$, the slope of secant line, m, approaches

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Since the derivative is related to the limiting case of the slope of the secant lines (the slope of the tangent line, m_t), we see that the derivative of $f(x)$ at $x = 2$ must be $\frac{1}{4}$