Calculus for the Life Sciences I

Lecture Notes – Velocity and Tangent

Joseph M. Mahaffy,
⟨mahaffy@math.sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://www-rohan.sdsu.edu/~jmahaffy

Spring 2013

Outline

Cats and Gravity

Cats and Gravity

Objects falling under the influence of gravity are important in classical differential Calculus

Sir Isaac Newton's work on gravity was a key step to the development of Calculus

Controversy as to whether Newton or Gottfried Leibnitz was the first to invent Calculus

Falling Cat 1

Falling Cats

- Cat have evolved to be one of the best mammalian predators
- Domestic cats have been shown to responsible for up to 60% of the deaths of songbirds in some communities
- They are adapted to hunting in trees
- Cats have a very flexible spine for hunting
- This flexibility allows them to rotate rapidly during a fall

Falling Cat 2

Falling Cats

- Humans have been fascinated by this ability of a cat to right itself
- Jared Diamond – Study of **Cats falling out of New York apartments**
  - Paradoxically the cats falling from the highest apartments actually fared better than ones falling from an intermediate height
  - The cat remains tense early in the fall
  - With greater heights the falling cat relaxes and spreads its legs to form a parachute
  - This slows its velocity a little and results in a more even impact
  - From intermediate heights, the cat basically achieves terminal velocity, but the tension causes increased likelihood of severe or fatal injuries

Falling Cat 3

Acceleration due to Gravity Consider a cat falling from a branch

- The early stages of the fall result from acceleration due to gravity
- **Newton's law of motion** says that **mass times acceleration is equal to the sum of all the forces** acting on an object
- **Velocity is the derivative of position**
- **Acceleration is the derivative of velocity**

Falling Cat 4

Suppose that a cat falls from a branch that is 16 feet high

The height of the cat satisfies the equation

$$h(t) = 16 - 16\,t^2$$

How long does this cat fall?

What is its velocity when it hits the ground?

Falling Cats 5

From the equation, the cat hits the ground when

$$h(t) = 16 - 16\,t^2 = 0$$

This occurs when $t = 1$

However, the velocity at $t = 1$ requires more work

We will show that the cat has a velocity,

$$v(1) = -32 \text{ ft/sec} \quad \text{(about 21.8 mph)}$$

Flight of a Ball

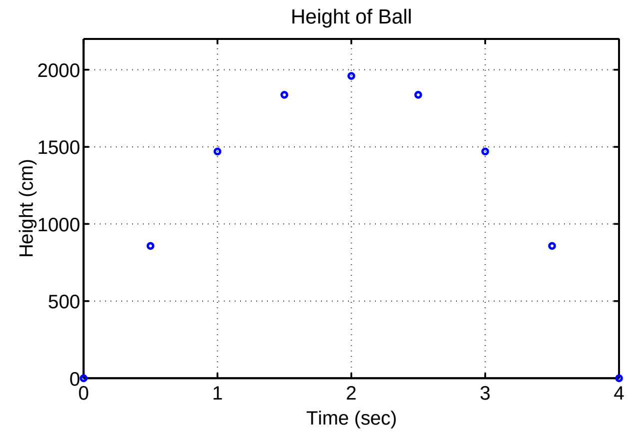
Consider a ball thrown vertically under the influence of gravity, ignoring air resistance

- The ball begins at ground level ($h(0) = 0$ cm)
- It is thrown vertically with an initial velocity, $v(0) = 1960$ cm/sec
- The acceleration of gravity is $g = 980$ cm/sec^2
- The height of the ball for any time t ($0 \le t \le 4$) is given by

$$h(t) = 1960\,t - 490\,t^2 = 0$$

Flight of a Ball

Graph of the height of a ball for $0 \le t \le 4$, showing position every 0.5 sec



Flight of a Ball

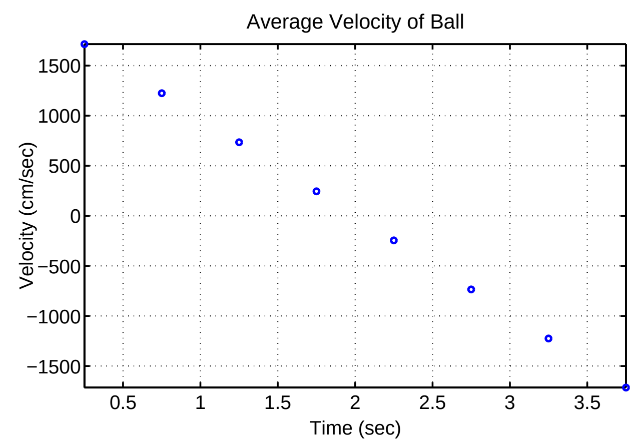
Compute the average velocity between each point on the graph

- The average velocity is the difference between the heights at two times divided by the length of the time period
- Associate the average velocity with the midpoint between each time interval

Height (t_1) $h(t_1)$	Height (t_2) $h(t_2)$	Average Time $t_a = (t_1 + t_2)/2$	Average Velocity $v(t_a) = \dfrac{h(t_2) - h(t_1)}{(t_2 - t_1)}$
$h(0) = 0$	$h(0.5) = 857.5$	$t_a = 0.5/2 = 0.25$	$v(0.25) = 1715$
$h(1.5) = 1837.5$	$h(2) = 1960$	$t_a = 1.75$	$v(1.75) = 245$
$h(3) = 1470$	$h(3.5) = 857.5$	$t_a = 3.25$	$v(3.25) = -1225$

Flight of a Ball

Graph of the velocity of a ball for $0 \le t \le 4$, showing velocity every 0.5 sec



Flight of Ball

The graph of the **height of the ball** as a function of time is a **parabola**

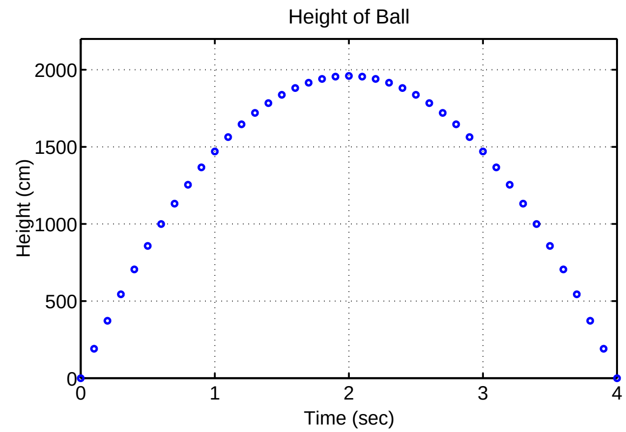
The graph of the **velocity of the ball** as a function of time is a **line**

The average velocity is zero when the ball reaches its maximum height

The **vertex of the parabola (maximum height of the ball)** is where the **velocity is zero (t-intercept)**

Flight of a Ball

Graph of the height of a ball for $0 \leq t \leq 4$, showing position every 0.1 sec

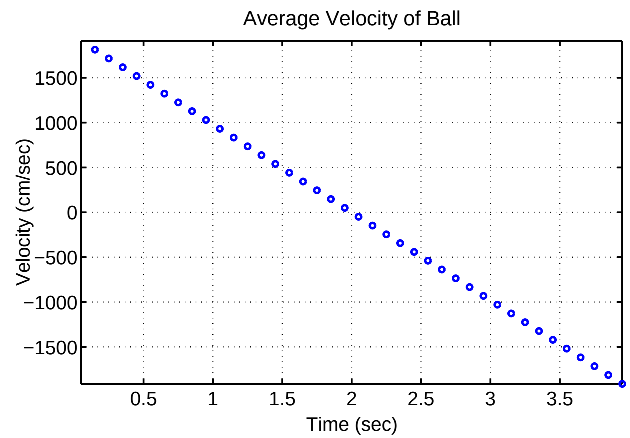


Flight of a Ball

- How does this affect the average velocity computation?
  - The distance between successive heights is now closer
  - But then the intervening time interval is also closer together
- The average velocity between $t_1 = 0.2$ and $t_2 = 0.3$ has $h(t_1) = 372.4$ cm and $h(t_2) = 543.9$ cm, so $v(0.25) = 1715$ cm/sec, the same as before

Flight of Ball

Graph of the velocity of a ball for $0 \leq t \leq 4$, showing velocity every 0.1 sec



Flight of Ball

- The average velocity data lie on the same straight line as before
$$v(t) = 1960 - 980\,t$$
- This straight line function is the **derivative** of the quadratic height function $h(t)$
- The calculation suggests that the derivative function is independent of the length of the time interval chosen
- This is specific to the **quadratic nature of the height function**
- Soon we will learn to take derivatives of more functions

Example – Flight of a Ball

A ball, which is thrown vertically with an initial velocity of 80 ft/sec and only the acceleration of gravity acting on the ball, satisfies the equation:
$$h(t) = 80\,t - 16\,t^2$$

Skip Example

- Sketch a graph of the height of the ball (in feet), $h(t)$, showing clearly the maximum height and when the ball hits the ground
- Find the average velocity of the ball between $t = 0$ and $t = 1$ and associate this velocity with $t = 0.5$
- Repeat this process for each second of the flight of the ball, then sketch a graph of the average velocity as a function of time, t

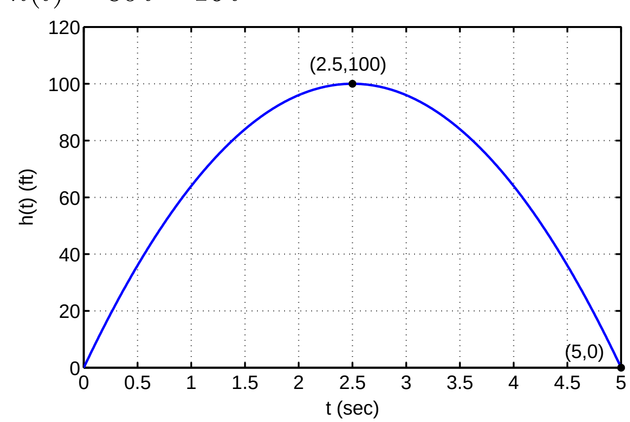
Example – Flight of a Ball

Graph of Flight of a Ball

- We write the equation
$$h(t) = -16\,t(t-5)$$
- $h(t)$ is a parabola with t-intercepts at $t = 0$ and $t = 5$
- The vertex or maximum height of the ball occurs at the midpoint between these intercepts or $t = 2.5$ with $h(2.5) = 100$ ft

Example – Flight of a Ball

Graph of $h(t) = 80\,t - 16\,t^2$



Example – Flight of a Ball 4

Average velocity for the Flight of a Ball

- The average velocity for the ball between $t = 0$ and $t = 1$ is given by

$$v_{ave}(0.5) = \frac{h(1) - h(0)}{1 - 0} = \frac{64 - 0}{1} = 64 \text{ ft/sec}$$

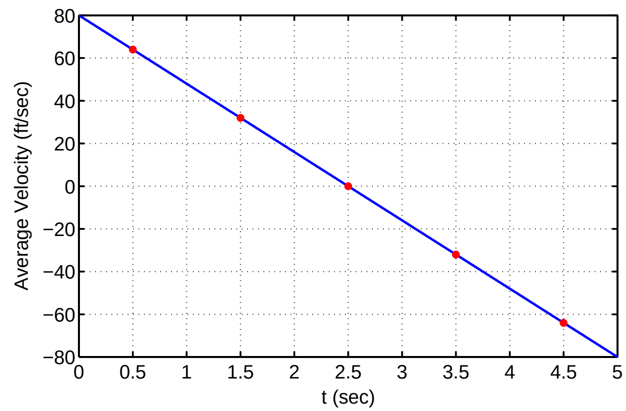
- The average velocities are computed between each pair of seconds from $t = 0$ to $t = 5$
-

t	0.5	1.5	2.5	3.5	4.5
v_{ave}	64	32	0	−32	−64

Example – Flight of a Ball 5

The average velocities fall on the straight line

$$v_{ave}(t) = 80 - 32\,t$$



Example – Leaping Salmon 1

- A river is dammed, and a salmon ladder is built to enable the salmon to bypass the dam and continue to travel upstream to spawn
- The vertical walls on the salmon ladder are 6 feet high
- The salmon has to leap vertically upwards over the wall
- The height of the salmon during its leap is given by

$$h(t) = v_0 t - 16\,t^2$$

Skip Example

Example – Leaping Salmon 2

- Let $v_0 = 20$ ft/sec. Sketch a graph of the height of the salmon $h(t)$, with time, showing clearly the maximum height and when the salmon can clear the wall
- Find the average velocity of the salmon between $t = 0$ and $t = 0.5$ and associate this velocity with $t = 0.25$
- Repeat this process for each half-second of the leaping salmon, then sketch a graph of the average velocity as a function of time, t
- Determine the minimum speed, v_0, that the salmon needs on exiting the water to climb the salmon ladder

Example – Leaping Salmon 3

Solution: The function $h(t)$ is a parabola,

$$h(t) = 20\,t - 16\,t^2 = 4\,t(5 - 4\,t)$$

- The t-intercepts are $t = 0$ and $t = 1.25$
- The vertex occurs at $(0.625, 6.25)$
- The salmon can clear the wall when $h(t) = 6$, so

$$20\,t - 16\,t^2 = 6 \quad \text{or} \quad 8\,t^2 - 10\,t + 3 = 0$$

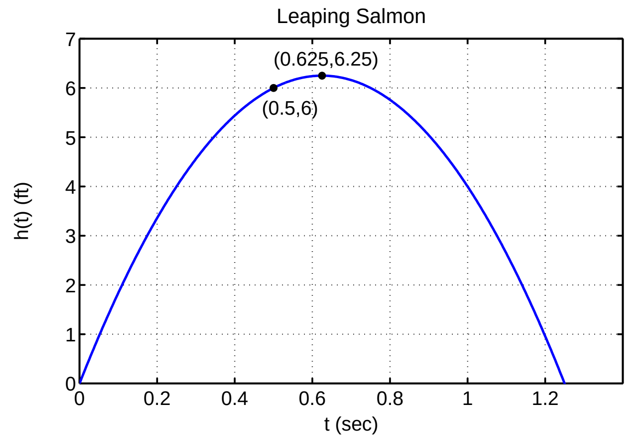
- This can be factored to give

$$(2\,t - 1)(4\,t - 3) = 0$$

- The salmon can clear the wall at any time $\frac{1}{2} < t < \frac{3}{4}$ sec

Example – Leaping Salmon 4

Graph of $h(t) = 20\,t - 16\,t^2$



Example – Leaping Salmon 5

Solution (cont):

- The average velocity of the salmon between $t = 0$ and $t = 0.5$ is given by,

$$v(0.25) = \frac{h(0.5) - h(0)}{0.5} = \frac{(20(0.5) - 16(0.5)^2) - 0}{0.5} = 12 \text{ ft/sec}$$

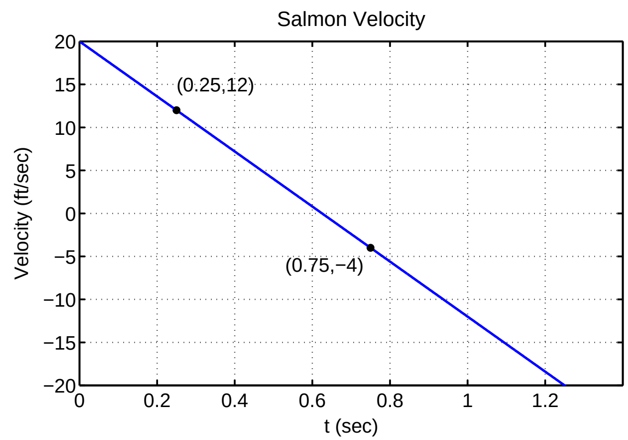
- The average velocity of the salmon between $t = 0.5$ and $t = 1$ is given by

$$v(0.75) = \frac{h(1) - h(0.5)}{0.5} = \frac{4 - 6}{0.5} = -4 \text{ ft/sec}$$

Example – Leaping Salmon 6

Graph of average velocity of the salmon satisfying

$$v_{ave}(t) = 20 - 32\,t$$



Example – Leaping Salmon

Solution (cont): The minimum speed, v_0, that the salmon needs to climb the fish ladder is the one that produces a maximum height of 6 ft

$$h(t) = v_0 t - 16\,t^2$$

- The $t-$value of the vertex occurs at
$$t = \frac{-v_0}{2(-16)} = \frac{v_0}{32}$$
- Since we want the vertex to be 6 ft,
$$h\left(\frac{v_0}{32}\right) = v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 = \frac{v_0^2}{64} = 6.$$
- $$v_0 = 8\sqrt{6} \approx 19.6 \text{ ft/sec}$$

SDSU

Secant Lines and Tangent Line

- The average velocity is the same calculation as the slope between the two data points of the height function
- The **slope of the secant line between two points on a curve**
- Geometrically, as the points on the curve get closer together, then the secant line approaches the tangent line
- The **tangent line** represents the best linear approximation to the curve near a given point
- Its slope is the derivative of the function at that point

SDSU

Secant Lines and Tangent Line

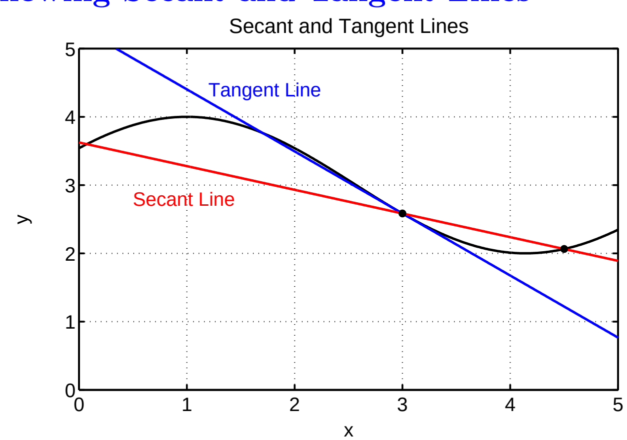
Definition: A **secant line** for a curve is a line that connect two points on the curve.

Definition: A **tangent line** for a curve is a line that touches the curve at exactly one point and provides the best approximation to the curve at that point.

SDSU

Secant Lines and Tangent Line

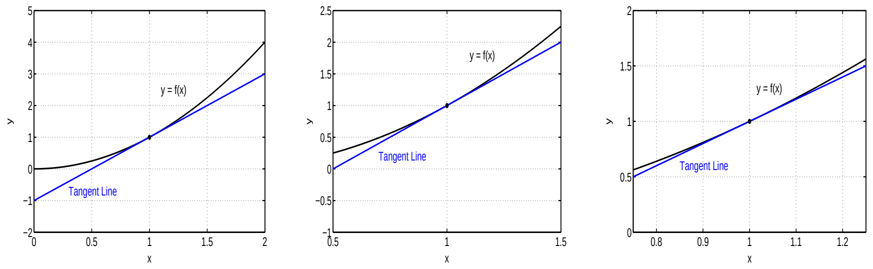
Graph showing Secant and Tangent Lines



SDSU

Tangent Line

A **tangent line** represents the best linear approximation to the curve near a given point



Example – $y = x^2$ 1

Consider the function

$$y = x^2$$

- Find the equation of the **tangent line** at the point (1,1) on the graph
- A **secant line** is found by taking two points on the curve and finding the equation of the line through those points
- Create a **sequence of secant lines that converge to the tangent line** by taking the two points closer and closer together

Example – $y = x^2$ 2

- Consider the secant line through the points (1,1) and (2,4)
- This line has a slope of $m = 3$, and its equation is

$$y = 3x - 2$$

- Consider the pair of points on the curve $y = x^2$, (1,1) and (1.5, 2.25)
- This line has a slope of $m = 2.5$, and its equation is

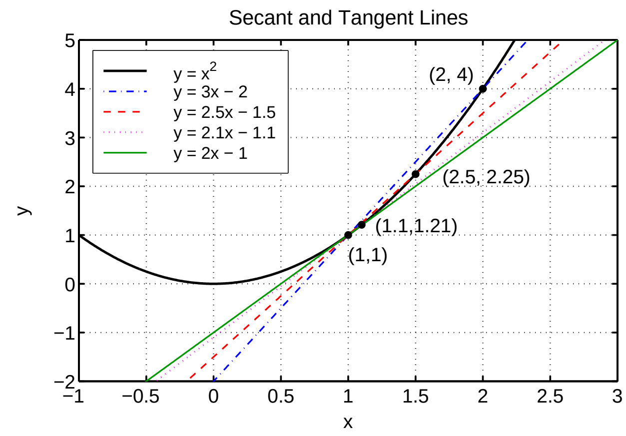
$$y = 2.5x - 1.5$$

- The secant line through the points (1,1) and (1.1, 1.21) has a slope of $m = 2.1$
- Its equation is

$$y = 2.1x - 1.1$$

Example – $y = x^2$ 3

Graph of $y = x^2$ with secant lines



Example – $y = x^2$

General secant line for $y = x^2$ at (1,1)

- Consider the x value $x = 1 + h$ for some small h
- The corresponding y value $y = (1+h)^2 = 1 + 2h + h^2$
- The slope of the secant line through this point and the point (1,1) is

$$m = \frac{(1 + 2h + h^2) - 1}{(1+h) - 1} = \frac{2h + h^2}{h} = 2 + h$$

- The formula for this secant line is

$$y = (2+h)x - (1+h)$$

Example – $y = x^2$

The general secant line for $y = x^2$ through (1,1) is

$$y = (2+h)x - (1+h)$$

- As h gets very small, the secant line gets very close to the tangent line
- Its not hard to see that the tangent line for $y = x^2$ at (1,1) is

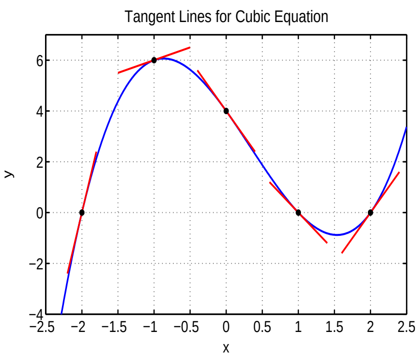
$$y = 2x - 1$$

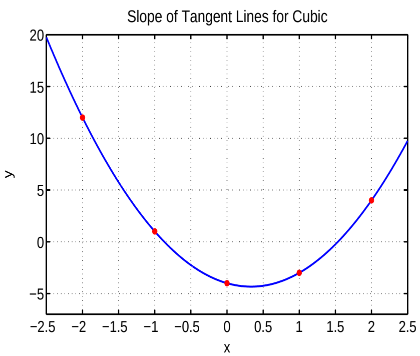
- The **slope of the tangent line is $m = 2$**
- The value of the **derivative of $y = x^2$ at $x = 1$**

Geometric view of Derivative

The geometric view of the tangent line is very easy to visualize

The graph on the left is $f(x)$ with tangent lines shown, while the graph on the right is the derivative of $f(x)$





Geometric view of Derivative

Several points of interest

- The graph on the left is a cubic function, while the graph of its derivative is a quadratic
- As you approach a maximum (or minimum) for the cubic function, the value of the derivative goes to zero and the sign of the derivative function changes
- This is an important application of the derivative

Example – Secant Lines 1

Consider the function

$$f(x) = x^2 - x$$

Skip Example

- Let all secant lines have the point, $x = 1$. Other points of the sequence have $x = 2$,$x = 1.5$,$x = 1.2$,$x = 1.1$, and $x = 1.01$
- Find the derivative of $f(x)$ at $x = 1$ by finding the slope of the tangent line at $x = 1$
- Graph $f(x)$, the tangent line, and the secant lines

SDSU

Example – Secant Lines 2

Solution: This example examines secant lines for

$$f(x) = x^2 - x$$

through the point $(1, 0)$

When $x = 2$, $f(2) = 2$, so the secant line has slope $m = 2$ and is given by

$$y = 2\,x - 2$$

For $x = 1.5$, two points on the secant line are $(1, 0)$ and $(1.5, 0.75)$, which gives the secant line

$$y = 1.5\,x - 1.5$$

SDSU

Example – Secant Lines 3

Solution (cont): Continuing the process:

When $x = 1.2$, two points on the secant line are $(1, 0)$ and $(1.2, 0.24)$, which gives the secant line

$$y = 1.2\,x - 1.2$$

For $x = 1.1$, two points on the secant line are $(1, 0)$ and $(1.1, 0.11)$, which gives the secant line

$$y = 1.1\,x - 1.1$$

For $x = 1.01$, two points on the secant line are $(1, 0)$ and $(1.01, 0.101)$, which gives the secant line

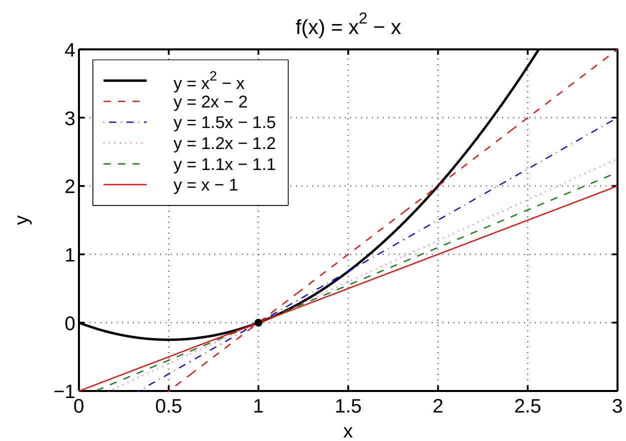
$$y = 1.01\,x - 1.01$$

SDSU

Example – Secant Lines 4

Solution (cont): The pattern in the sequence easily gives the tangent line

$$y = x - 1$$



SDSU

Example – Secant Lines 5

Solution (cont): Since the tangent line has slope $m = 1$, the derivative of $f(x) = x^2 - x$ at $x = 1$ is **1**

Since patterns cannot always be recognizable, we need a better way to compute the derivative

Example – Secant Lines 6

Solution (cont): Let's find the slope of the secant line through the points

$$(1, f(1)) = (1, 0) \quad \text{and} \quad (1+h, f(1+h))$$

Since $f(1+h) = (1+h)^2 - (1+h) = h^2 + h$, the slope of the secant line is

$$m(h) = \frac{(h^2+h) - 0}{(1+h) - 1} = \frac{h^2+h}{h} = 1 + h$$

As $h \to 0$, $m(h) \to 1$

It follows that the slope of the tangent line is **1**, which is the derivative of $f(x)$ at $x = 1$

Velocity of Falling Cat

Return to the cat falling from a 16 ft tree limb, where

$$h(t) = 16 - 16\,t^2$$

Recall the cat hits the ground at $t = 1$ sec

We find the general secant line between $t = 1$ and $t = 1 + z$, which relates to the **Average Velocity** near $t = 1$

Since $h(1+z) = 16 - 16(1+z)^2 = -32\,z - 16\,z^2$

$$v_{ave} = \frac{h(1+z) - h(1)}{(1+z) - 1} = \frac{-32z - 16z^2}{z} = -32 - 16\,z$$

As $z \to 0$, $v_{ave} \to -32$, so the cat hits the ground at a velocity of -32 ft/sec ($\simeq 21.8$ mph)

Example – Square Root Function 1

Consider the function

$$f(x) = \sqrt{x+2}$$

- Find the slope of the secant line through the points $(2, f(2))$ and $(2+h, f(2+h))$
- Let h get small and determine the slope of the tangent line through $(2, 2)$, which gives the value of the derivative of $f(x)$ at $x = 2$

Example – Square Root Function

Solution: The slope of the secant line is

$$\begin{aligned} m(h) &= \frac{f(2+h)-f(2)}{(2+h)-2} \\ &= \frac{\sqrt{2+h+2}-\sqrt{2+2}}{h} = \frac{\sqrt{4+h}-2}{h} \\ &= \left(\frac{\sqrt{4+h}-2}{h}\right)\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right) \\ &= \frac{4+h-4}{h(\sqrt{4+h}+2)} \\ &= \frac{1}{\sqrt{4+h}+2} \end{aligned}$$

Example – Square Root Function

Solution (cont): The slope of the secant line is

$$m(h) = \frac{1}{\sqrt{4+h}+2}$$

In the formula above, as $h \to 0$, the slope of secant line, m, approaches

$$m_t = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

Since the derivative is related to the limiting case of the slope of the secant lines (the slope of the tangent line, m_t), we see that the derivative of $f(x)$ at $x = 2$ must be $\frac{1}{4}$