Calculus for the Life Sciences I

Lecture Notes – Rules of Differentiation

Joseph M. Mahaffy,
⟨mahaffy@math.sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://www-rohan.sdsu.edu/∼jmahaffy

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Outline

Introduction

- The previous section showed the definition of a derivative

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- Applications to polynomials

Applications with Power Law

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- Obtained data from Altman and Dittmer for the pulse and weight of mammals
- The pulse, P, as a function of the weight, w, are approximated by the relationship

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- The pulse is in beats/min, and the weight is in kilograms

Pulse and Weight

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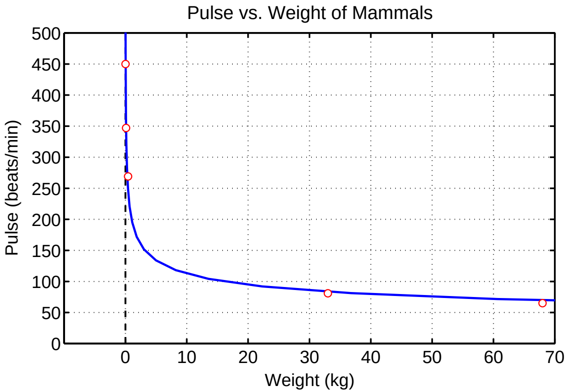

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- Can one quantify how fast the pulse rate changes as a function of weight?
- For small animals the pulse rate changes more rapidly than for large animals
- The derivative of this allometric or power law model provides more details on the rate of change in pulse rate as a function of weight

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- A model of this sort is important for obtaining information about biodiversity

Biodiversity and Area

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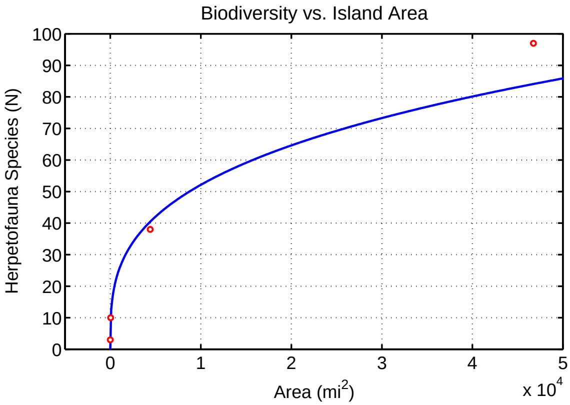

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- Can we use this model to determine the rate of change of numbers of species with respect to a given increase in area?

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- Again the derivative is used to help quantify the rate of change of the dependent variable, N, with respect to the independent variable, A

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- We will use these notations interchangeably

Power Rule

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The **power rule for differentiation** is given by the formula

$$\frac{d(x^n)}{dx} = nx^{n-1}, \qquad \text{for} \quad n \neq 0$$

Examples of the Power Rule

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Since $n = 0$, the power rule does not apply

However, we know the **derivative** of a **constant** is

$$f'(x) = 0$$

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The derivative of $f(x)$ satisfies

$$f\,'(x) = \frac{d}{dx}(2\,x^{1/2}) + \frac{d}{dx}(x^4) = x^{-1/2} + 4\,x^3$$

Linear Approximation

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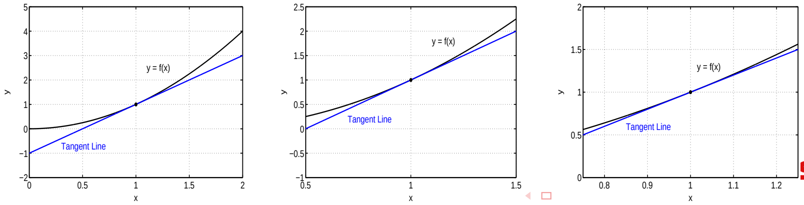
- Recall that the **tangent line** gives a **linear approximation** of a function near the point of tangency



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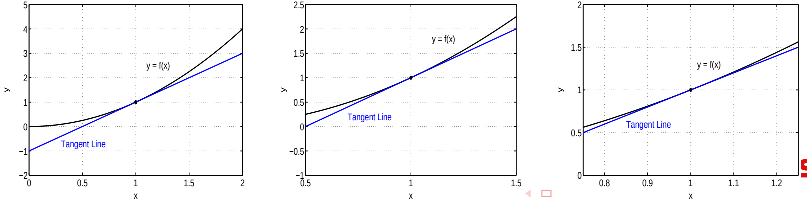

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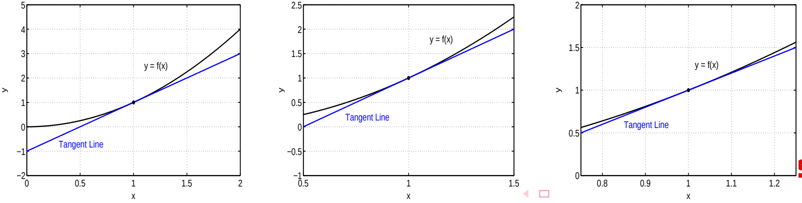

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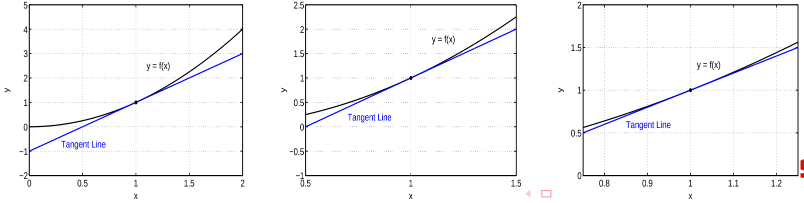

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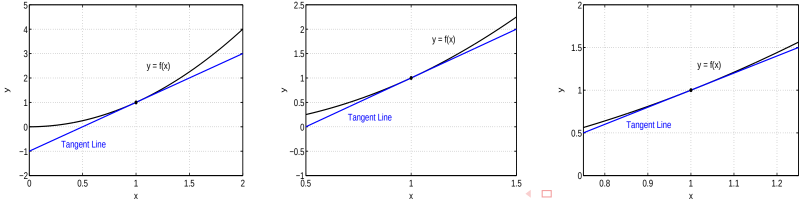
- This technique is often used in **Error Analysis**



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The negative sign shows the decrease in the pulse rate with increasing weight

Pulse and Weight Example 2

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- The power law of differentiation gives
$$\frac{dP}{dt} = -50\,w^{-5/4}$$
- The derivative at $w = 16$ is
$$P'(16) = -50(16)^{-5/4} = -\frac{50}{32} \approx -1.56$$

Pulse and Weight Example 3

Example for Linear Approximation (cont):

- The tangent line approximation, $P_L(w)$, near $w = 16$ is

$$P_L(w) = -\frac{50}{32}(w - 16) + 100$$

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Example for Linear Approximation (cont):

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- It follows that a 17 kg animal should have a pulse near

$$P_L(17) = -\frac{50}{32}(1) + 100 \approx 98.44 \text{ beats/min}$$

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- Note that the **Allometric model** gives

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- The increase gets smaller with increasing island area, since the area has the power $-2/3$, which puts the area in the denominator of this expression for the derivative

Skip Linear Approximation

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- An island with 1000 sq mi by the allometric model would have approximately 30 species (since $N(1000) = 3(1000)^{1/3} = 30$)

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- The derivative at $A = 1000$ is

$$N'(1000) = (1000)^{-2/3} = 0.01$$

Biodiversity Example 3

Example for Linear Approximation (cont):

- The tangent line approximation, $N_L(A)$, near $A = 1000$ is

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- It follows that an island with an area of 950 sq mi should have approximately

$$N_L(950) = 0.01(950 - 1000) + 30 = 29.5 \text{ species}$$

- Note that the **Allometric model** gives

$$N(950) = 3(950)^{1/3} = 29.49 \text{ species}$$

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Assume the only acceleration is due to gravity, g and air resistance ignored

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  - Each term has a scalar multiple
  - Power rule can be applied to the t and t^2 terms

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From our rules above, the derivative is

$$f\,'(x) = 2\,x - 6\,x^{-3} - 4\,x^{-1/2}$$

Velocity of a Ball 1

Example: A ball, thrown vertically from a platform without air resistance, satisfies the equation

$$h(t) = 80 + 64\,t - 16\,t^2$$

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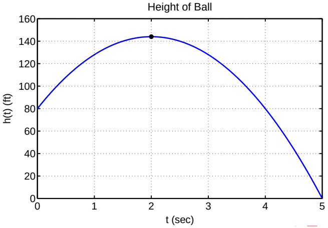

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- The ball hits the ground with velocity $v(5) = -96$ ft/sec

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  - Crowding (lack of space to reproduce)
  - Lack of resources (limited food supply)
  - Build up of waste (toxicity)

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- Population reaches an equilibrium, the **carrying capacity**

Logistic Growth Model 3

Example of Logistic Growth Function: Suppose a culture of yeast has the growth function

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where P is the density of yeast $(\times 1000/\text{cc})$

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  - The Malthusian growth rate $r = 0.1$
  - The parameter $M = 500$

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  - The **extinction** equilibrium, $P_e = 0$
  - The **carrying capacity**, $P_e = M = 500$

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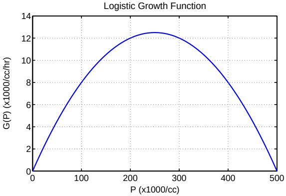

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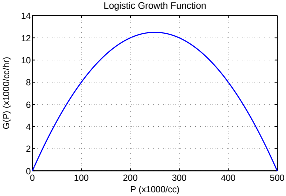

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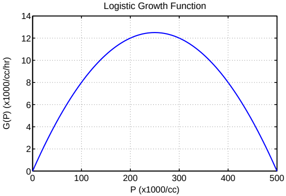

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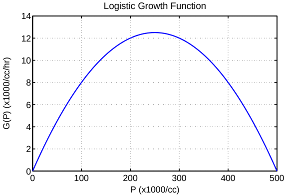

- This model gives **equilibria** at $P_e = 0$ and $P_e = 500$ $(\times 1000)$ yeast/cc
- Maximum population growth occurs at $P_v = 250$ $(\times 1000)$ yeast/cc
- Since $G(250) = 12.5$, when the **density of yeast** is 250 $(\times 1000)$ yeast/cc, the maximum production is 12.5 $(\times 1000)$ yeast/cc/hr

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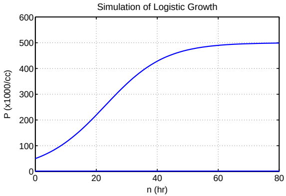

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- Simulation shows the population approaching the **carrying capacity** of 500 and the maximum growth near $n = 25$