Calculus for the Life Sciences I

Lecture Notes – Quadratic Equations and Functions

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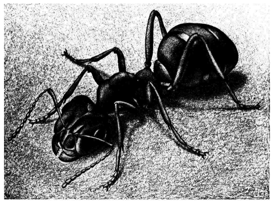
Outline

Weak Acids

- Many of the organic acids found in biological applications are weak acids
- Weak acid chemistry is important in preparing buffer solutions for laboratory cultures

Formic Acid



Ants

- Formic acid (HCOOH) is a relatively strong weak acid that ants use as a defense
- The strength of this acid makes the ants very unpalatable to predators

Acid Chemistry

The **Chemistry of Dissociation** for formic acid:

$$\mathrm{HCOOH} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{H^+ + HCOO^-}.$$

- Each acid has a distinct equilibrium constant K_a that depends on the properties of the acid and the temperature of the solution
- For formic acid, $K_a = 1.77 \times 10^{-4}$
- Let $[X]$ denote the concentration of chemical species X
- Formic acid is in equilibrium, when:

$$K_a = \frac{[H^+][HCOO^-]}{[HCOOH]}$$

Concentration of $[H^+]$ 1

Based on K_a and amount of formic acid, we want to find the concentation of $[H^+]$

- If formic acid is added to water, then $[H^+] = [HCOO^-]$
- If x is the normality of the solution, then $x = [HCOOH] + [HCOO^-]$
- It follows that $[HCOOH] = x - [H^+]$
- Thus,

$$K_a = \frac{[H^+][H^+]}{x - [H^+]}$$

Concentration of $[H^+]$ 2

The previous equation is written

$$[H^+]^2 + K_a[H^+] - K_a x = 0$$

This is a quadratic equation in $[H^+]$ and is easily solved using the quadratic formula

$$[H^+] = \frac{1}{2}\left(-K_a + \sqrt{K_a^2 + 4K_a x}\right)$$

Only the positive solution is taken to make physical sense

Example for $[H^+]$

Find the concentration of $[H^+]$ for a 0.1N solution of formic acid

Solution: Formic acid has $K_a = 1.77 \times 10^{-4}$, and a 0.1N solution of formic acid gives $x = 0.1$

The equation above gives

$$[H^+] = \frac{1}{2}\left(-0.000177 + \sqrt{(0.000177)^2 + 4(0.000177)(0.1)}\right)$$

or

$$[H^+] = 0.00412$$

Since pH is defined to be $-\log_{10}[H^+]$, this solution has a pH of 2.385

Review of Quadratic Equations

Quadratic Equation: The general quadratic equation is

$$ax^2 + bx + c = 0$$

Three methods for solving quadratics:

1. Factoring the equation
2. The **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Completing the Square

Example of Factoring a Quadratic Equation

Consider the quadratic equation:

$$x^2 + x - 6 = 0$$

Find the values of x that satisfy this equation.

Skip Example

Solution: This equation is easily factored

$$(x + 3)(x - 2) = 0$$

Thus,

$$x = -3 \quad \text{and} \quad x = 2$$

Example of the Quadratic Formula

Consider the quadratic equation:

$$x^2 + 2x - 2 = 0$$

Find the values of x that satisfy this equation.

Skip Example

Solution: This equation needs the quadratic formula

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)} = -1 \pm \sqrt{3}$$

or

$$x = -2.732 \quad \text{and} \quad x = 0.732$$

Example with Complex Roots

Consider the quadratic equation:

$$x^2 - 4x + 5 = 0$$

Find the values of x that satisfy this equation.

Skip Example

Solution: We solve this by **completing the square**

Rewrite the equation

$$x^2 - 4x + 4 = -1$$

$$(x - 2)^2 = -1 \quad \text{or} \quad x - 2 = \pm\sqrt{-1} = \pm i$$

This has no real solution, only the complex solution

$$x = 2 \pm i$$

Quadratic Function

The general form of the Quadratic Function is

$$f(x) = ax^2 + bx + c,$$

where $a \neq 0$ and b and c are arbitrary.

The graph of

$$y = f(x)$$

produces a parabola

Vertex

Write the quadratic function (recall completing the squares)

$$y = a(x-h)^2 + k$$

The **Vertex of the Parabola** is the point

$$(x_v, y_v) = (h, k)$$

The parameter a determines the direction the parabola opens

- If $a > 0$, then the parabola opens upward
- If $a < 0$, then the parabola opens downward
- As $|a|$ increases the parabola narrows

Finding the Vertex

Given the quadratic function

$$y = ax^2 + bx + c$$

There are three common methods of finding the **vertex**

- The x-value is $x = -\frac{b}{2a}$
- The midpoint between the x-intercepts (if they exist)
- Completing the square

Example of Line and Parabola 1

Consider the functions

$$f_1(x) = 3 - 2x \quad \text{and} \quad f_2 = x^2 - x - 9$$

Skip Example

- Find the x and y intercepts of both functions
- Find the slope of the line
- Find the vertex of the parabola
- Find the points of intersection
- Graph the two functions

Example of Line and Parabola 2

Solution: The line

$$f_1(x) = 3 - 2x$$

- Has y-intercept $y = 3$
- Has x-intercept $x = \frac{3}{2}$
- Has slope $m = -2$

Example of Line and Parabola 3

Solution (cont): The parabola

$$f_2 = x^2 - x - 9$$

- Has y-intercept $y = -9$, since $f_2(0) = -9$
- By quadratic formula the x-intercepts satisfy

$$x = \frac{1 \pm \sqrt{37}}{2} \quad \text{or} \quad x \approx -2.541, 3.541$$

- Vertex satisfies $x = \frac{1}{2}$ and $y = -\frac{37}{4}$

Example of Line and Parabola 4

Solution (cont): The points of intersection of

$$f_1(x) = 3 - 2x \quad \text{and} \quad f_2 = x^2 - x - 9$$

Find the points of intersection by setting the equations equal to each other

$$3 - 2x = x^2 - x - 9 \quad \text{or} \quad x^2 + x - 12 = 0$$

Factoring

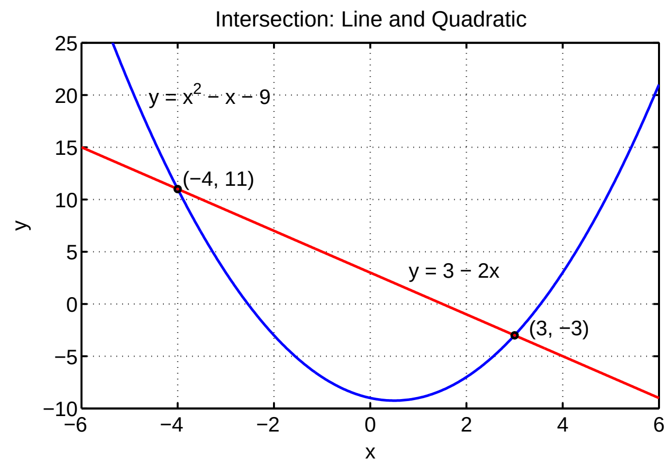
$$(x+4)(x-3) = 0 \quad \text{or} \quad x = -4, 3$$

Points of intersection are

$$(x_1, y_1) = (-4, 11) \quad \text{or} \quad (x_2, y_2) = (3, -3)$$

Example of Line and Parabola 5

Solution (cont): Graph of the functions



Height of a Ball 1

A ball is thrown vertically with a velocity of 32 ft/sec from ground level ($h = 0$). The height of the ball satisfies the equation:

$$h(t) = 32\,t - 16\,t^2$$

Skip Example

- Sketch a graph of $h(t)$ vs. t
- Find the maximum height of the ball
- Determine when the ball hits the ground

Example of Line and Parabola 2

Solution: Factoring

$$h(t) = 32\,t - 16\,t^2 = -16\,t(t-2)$$

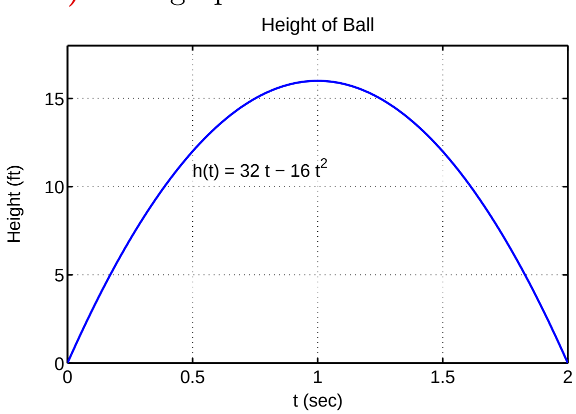
This gives t-intercepts of $t = 0$ and 2

The midpoint between the intercepts is $t = 1$

Thus, the vertex is $t_v = 1$, and $h(1) = 16$

Example of Line and Parabola 3

Solution (cont): The graph is



- The maximum height of the ball is 16 ft
- The ball hits the ground at $t = 2$ sec

Lambert-Beer Law 1

Concentration and Absorbance

- A spectrophotometer uses the Lambert-Beer law to determine the concentration of a sample (c) based on the absorbance of the sample (A)
- The ion dichromate forms an orange/yellow that has a maximum absorbance at 350 nm and is often used in oxidation/reduction reactions
- The **Lambert-Beer law** for the concentration of a sample from the absorbance satisfies the linear model

$$c = mA$$

where m is the slope of the line (assuming the spectrophotmeter is initially zeroed)

Lambert-Beer Law

Spectrophotometer data for an redox reaction

- Data collected on some known samples

A	0.12	0.32	0.50	0.665
c (mM)	0.05	0.14	0.21	0.30

- Determine the quadratic function $J(m)$ that measures the sum of the squares of the error of the linear model to the data
- Sketch a graph of $J(m)$ and find the vertex of this quadratic function
- Sketch a graph of the data and the line that best fits the data
- Use this model to determine the concentration of two unknown samples that have absorbances of $A = 0.45$ and 0.62

Lambert-Beer Law

Solution: Given the linear model $c = mA$, the sum of square errors satisfies

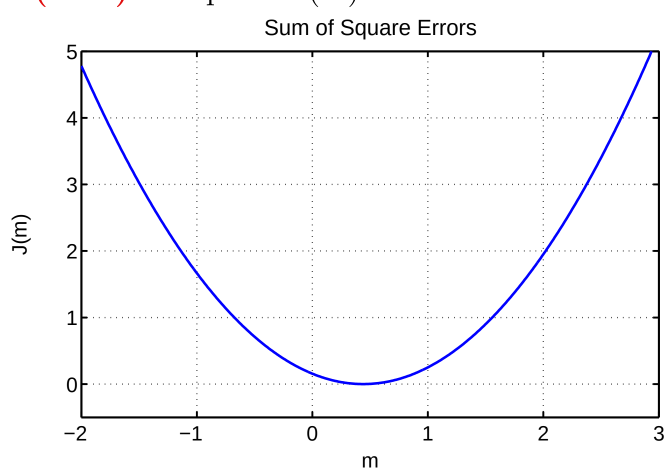
$$\begin{aligned} J(m) &= e_1^2 + e_2^2 + e_3^2 + e_4^2 \\ &= (0.05 - 0.12m)^2 + (0.14 - 0.32m)^2 + (0.21 - 0.50m)^2 + (0.30 - 0.66m)^2 \\ &= 0.8024m^2 - 0.7076m + 0.1562 \end{aligned}$$

The vertex has $m_v = \frac{0.7076}{2(0.8024)}$, so

$$(m_v, J(m_v)) = (0.44093, 0.00019995)$$

Lambert-Beer Law

Solution (cont): Graph of $J(m)$



Lambert-Beer Law

Solution (cont): Since the vertex has $m_v = 0.44093$, the **best linear model** is

$$c = 0.441\, m$$

For an absorbance $A = 0.45$

$$c(0.45) = 0.441(0.45) = 0.198$$

The best model predicts a concentration of 0.198 nM

For an absorbance $A = 0.62$

$$c(0.62) = 0.441(0.62) = 0.273$$

The best model predicts a concentration of 0.273 nM

Lambert-Beer Law

Solution (cont): Graph of Best Linear Model and Data