Calculus for the Life Sciences I

Lecture Notes – Nonlinear Dynamical Systems

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Outline

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Introduction

Discrete Growth Models

- The Discrete Malthusian growth model shows exponential growth
- Most animal populations grow exponentially soon after settling
- With population growth, crowding pressure decreases the growth rate
  - Space and resource limitation
  - Toxic build up

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Yeast Study   1

Growing Culture of Yeast: Classic study by Carlson in 1913

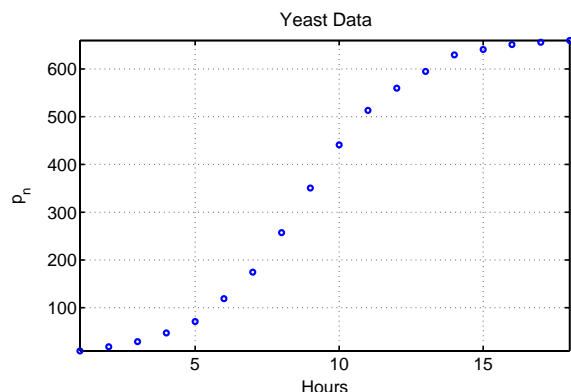| Time	Population	Time	Population	Time	Population
1	9.6	7	174.6	13	594.8
2	18.3	8	257.3	14	629.4
3	29.0	9	350.7	15	640.8
4	47.2	10	441.0	16	651.1
5	71.1	11	513.3	17	655.9
6	119.1	12	559.7	18	659.6

These data show a classic **S-shape curve**

[1] T. Carlson Über Geschwindigkeit und Grösse der Hefevermehrung in Würze. *Biochem. Z.* (1913) **57**, 313–334

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Discrete Logistic Growth Model
Qualitative Analysis of Logistic Growth Model
Cobwebbing

Introduction
Yeast Study
Discrete Dynamical Models

Yeast Study　　2

Carlson (1913) Yeast data: Classic **S-shape curve** with initial accelerating growth, then eventually saturation

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Discrete Dynamical Models

Discrete Growth Models　　1

Discrete Dynamical Growth Model

There are two standard forms for **discrete population models**

One form uses a **growth function**, $G(p_n)$

$$p_{n+1} = p_n + G(p_n)$$

The population at the next time interval $(n + 1)$ equals the population at the current time interval (n) plus the net growth of the current population, $G(p_n)$

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Discrete Growth Models　　2

Discrete Dynamical Model with Updating Function

A more general form satisfies

$$p_{n+1} = F(p_n)$$

- An **iterative map** – the population at the $(n+1)^{st}$ generation depends on the population at the nth generation
- The function $F(p)$ is called the **updating function**
- The graph of the updating function
  - The $(n+1)^{st}$ generation is on the vertical axis
  - The nth generation is on the horizontal axis
  - Usually want **identity map** to find **equilibria**

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Logistic Growth Model　　1

Logistic Growth Model
- Malthusian growth uses a linear updating function and grows exponentially without bound
- Most populations have a decreasing growth rate due to crowding effects
- Easiest form is to insert a quadratic term (negative) to the updating function
- This is the **Logistic Growth model**

$$p_{n+1} = p_n + rp_n \left(1 - \frac{p_n}{M}\right)$$

- This equation has the Malthusian growth model with the additional term $-rp_n^2/M$
- The parameter M is called the **carrying capacity** of the population

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Logistic Growth Model   2

Behavior of the Logistic Growth Model

- The Logistic growth model shows complicated dynamics – shown by ecologist May (1974)
- There is **no exact solution** to this discrete dynamical system
- Given the **Logistic Growth model**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

  - There are **equilibria** at 0 and M
  - The parameter r has restricted values ($r < 3$) with more complex behavior for higher values of r
- Numerous applets available on the web to view behavior   **SDSU**

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Yeast Study   1

Logistic Growth Model for Carlson Yeast Study

- **Logistic Growth model** has form
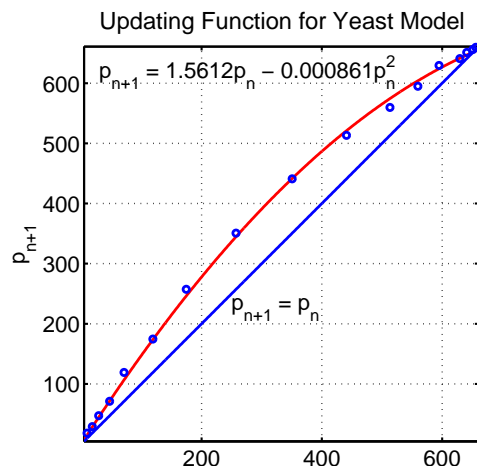
$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

- Use successive data values to obtain p_{n+1} and p_n
- The first two points are $(9.6, 18.3)$ and $(18.3, 29.0)$ with others found similarly
- The graph of the data is fit with the best quadratic passing through the origin   **SDSU**

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Yeast Study   2

Updating Function: Graph of best fitting **quadratic** through the origin of data, p_{n+1} vs p_n, and the identity function

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Yeast Study   3

- Recall the **logistic growth model** has the form

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

- The best fitting model to the yeast data is

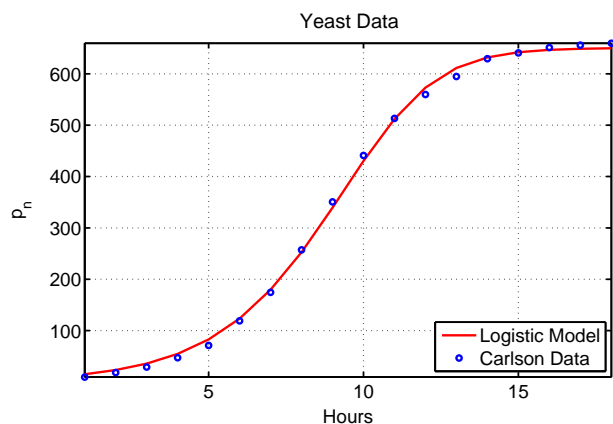$$p_{n+1} = 1.5612\,p_n - 0.000861\,p_n^2$$

- It follows that $r = 0.5612$ and $M = 650.4$ **SDSU**

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Yeast Study 4

Simulation: The model is easily simulated and by varying the initial population to $p_1 = 15.0$, a best fit to the data is found

Yeast Data

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Equilibria
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Equilibria

Consider the **general discrete dynamical model:**

$$p_{n+1} = F(p_n)$$

Study the **qualitative behavior of discrete dynamical equations**

- The **first step in any analysis** is finding **equilibria**
- This is simply an **algebraic equation**
- An equilibrium point of a discrete dynamical system is where there is no change in the variable from one iteration to the next
- Mathematically, $p_e = F(p_e)$
- Geometrically, this is when $F(p)$ crosses the **identity map**

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Equilibria for Logistic Growth Model

Consider the **logistic growth model:**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

If $r > 0$, then equilibria satisfy

$$p_e = p_e + rp_e\left(1 - \frac{p_e}{M}\right)$$

$$rp_e\left(1 - \frac{p_e}{M}\right) = 0$$

Thus, $p_e = 0$ or $p_e = M$

The equilibria for the Logistic growth model are either

- The **trivial solution** $p_e = 0$ (no population) or
- The **carrying capacity** $p_e = M$

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Logistic Growth Model Simulation 1

Consider the **logistic growth model:**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

Let $p_0 = 50$, $M = 1000$, and $r = 0.5$

Model Simulation r = 0.5

Simulation monotonically approaches carrying capacity

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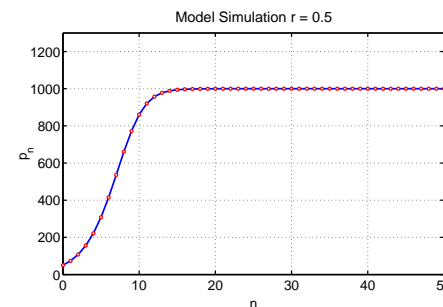Examples of Logistic Growth
U. S. Population Models

Logistic Growth Model Simulation 2

Consider the **logistic growth model:**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

Let $p_0 = 50$, $M = 1000$, and $r = 1.8$

Model Simulation r = 1.8

Simulation oscillates, but approaches carrying capacity

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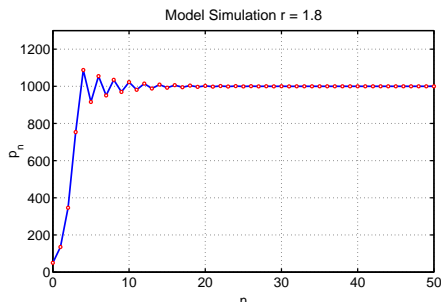Examples of Logistic Growth
U. S. Population Models

Logistic Growth Model Simulation 3

Consider the **logistic growth model:**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

Let $p_0 = 50$, $M = 1000$, and $r = 2.3$

Model Simulation r = 2.3

Simulation oscillates with period 2 about carrying capacity

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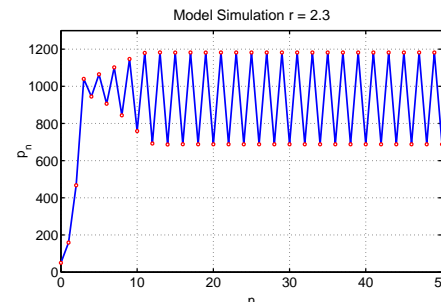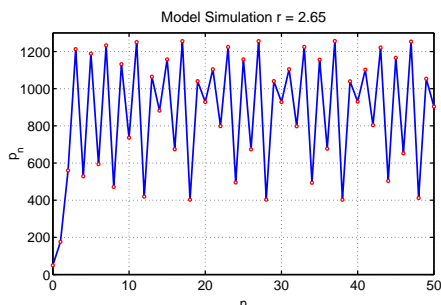Examples of Logistic Growth
U. S. Population Models

Logistic Growth Model Simulation 4

Consider the **logistic growth model:**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

Let $p_0 = 50$, $M = 1000$, and $r = 2.65$

Model Simulation r = 2.65

Simulation is chaotic with unpredictable results

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Stability of Logistic Growth Model 1

Stability of Logistic Growth Model

- Equilibria are easy to find, but behavior of the model varies dramatically as shown by simulations above
- There are mathematical tools that help predict some of these behaviors
- The discrete logistic growth model is

$$p_{n+1} = f(p_n) = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

- The derivative of the function $f(p)$ is valuable for determining the behavior of the discrete dynamical system near an equilibrium point

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Stability of Logistic Growth Model   2

- The **Equilibria** are

$$p_e = 0 \quad \text{and} \quad p_e = M$$

- The **derivative** of $f(p) = (1+r)p - rp^2/M$ is

$$f'(p) = 1 + r - \frac{2rp}{M}$$

- Evaluation of the derivative at the equilibria gives some information about the **behavior of the discrete dynamical model**

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Stability of Logistic Growth Model   3

Consider the **Trivial Equilibrium**, $p_e = 0$

- Since the **derivative** is

$$f'(p) = 1 + r - \frac{2rp}{M}$$

- At $p_e = 0$, the derivative satisfies

$$f'(0) = 1 + r$$

- r positive always results in solutions growing away from this equilibrium
- When the population is small, there are plenty of resources and the population grows (exponentially)
- Near $p_e = 0$ solutions behave like Malthusian growth

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Stability of Logistic Growth Model   4

Consider the **Carrying Capacity Equilibrium**, $p_e = M$

- Since the **derivative** is

$$f'(p) = 1 + r - \frac{2rp}{M}$$

- At $p_e = M$, the derivative satisfies

$$f'(M) = 1 - r$$

- There are several possible behaviors of the solution near the carrying capacity equilibrium

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Behavior of Discrete Dynamical Models

- If $f'(p_e) > 1$
  - Solutions of the discrete dynamical model grow away from the equilibrium (monotonically)
  - **The equilibrium is unstable**
- If $0 < f'(p_e) < 1$
  - Solutions of the discrete dynamical model approach the equilibrium (monotonically)
  - **The equilibrium is stable**
- If $-1 < f'(p_e) < 0$
  - Solutions of the discrete dynamical model oscillate about the equilibrium and approach it
  - **The equilibrium is stable**
- If $f'(p_e) < -1$
  - Solutions of the discrete dynamical model oscillate about the equilibrium but move away from it
  - **The equilibrium is unstable**

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Behavior of the Logistic Growth Model

Behavior of Logistic Growth Model near $p_e = M$

- If $0 < r < 1$, then the solution of the discrete logistic model **monotonically approaches the equilibrium**, $p_e = M$, which was observed for the experiment with the yeast
- If $1 < r < 2$, then the solution of the discrete logistic model **oscillates about the equilibrium**, $p_e = M$, but the solution **asymptotically approaches** this equilibrium
- If $2 < r < 3$, then the solution of the discrete logistic model **oscillates about the equilibrium**, $p_e = M$, but the solution **grows away from** this equilibrium
- $r > 3$ results in negative solutions

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Example 1 of the Logistic Growth Model   1

Example 1: Consider the **discrete logistic growth model**

$$p_{n+1} = f_1(p_n) = 1.3\,p_n - 0.0001\,p_n^2$$

Skip Example

- Find all the equilibria for this model
- Determine the behavior of the solution near these equilibria
- Sketch a graph of the updating function and the identity map $p_{n+1} = p_n$
- Simulate the model, starting $p_0 = 100$ for 50 iterations

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Example 1 of the Logistic Growth Model   2

Solution: For the **discrete logistic growth model**

$$p_{n+1} = 1.3\,p_n - 0.0001\,p_n^2$$

the equilibria are found by substituting $p_e = p_n = p_{n+1}$

Thus,

$$
\begin{aligned}
p_e &= 1.3\,p_e - 0.0001\,p_e^2 \\
0 &= 0.3\,p_e - 0.0001\,p_e^2 = p_e(0.3 - 0.0001\,p_e)
\end{aligned}
$$

The equilibria satisfy

$$p_e = 0$$

and

$$0.3 - 0.0001\,p_e = 0 \quad \text{or} \quad p_e = 3000$$

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Example 1 of the Logistic Growth Model   3

Solution (cont): For $f_1(p) = 1.3\,p - 0.0001\,p^2$, the derivative satisfies

$$f_1'(p) = 1.3 - 0.0002\,p$$

At $p_e = 0$

$$f_1'(0) = 1.3 > 1$$

The solution monotonically grows away from this equilibrium, as expected

At $p_e = 3000$

$$f_1'(3000) = 1.3 - 0.6 = 0.7 < 1$$

The solution monotonically approaches this equilibrium
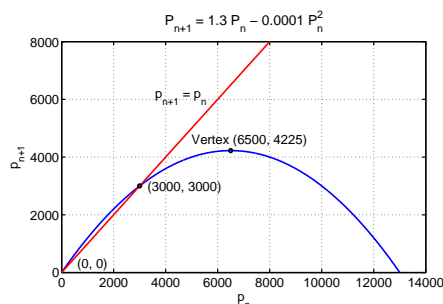This equilibrium is **stable**

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Example 1 of the Logistic Growth Model 4

Graphing the updating function

- The p-intercepts are 0 and 13,000
- The vertex is at (6500, 4225)
- Below is graph of updating function and identity map with significant points
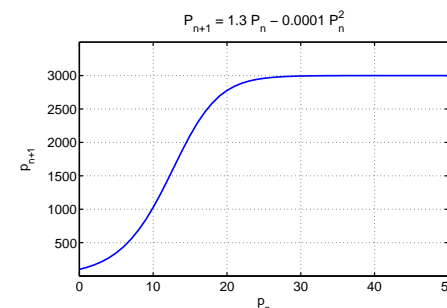

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Example 1 of the Logistic Growth Model 5

Simulation of

$$p_{n+1} = 1.3\, p_n - 0.0001\, p_n$$

with $p_0 = 100$ for 50 iterations

Shows classic S-curve of population growth

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Example 2 of the Logistic Growth Model 1

Example 2: Consider the **discrete logistic growth model**

$$p_{n+1} = f_2(p_n) = 2.7\, p_n - 0.0001\, p_n^2$$

Skip Example

- Find all the equilibria for this model
- Determine the behavior of the solution near these equilibria
- Sketch a graph of the updating function and the identity map $p_{n+1} = p_n$
- Simulate the model, starting $p_0 = 100$ for 50 iterations

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Example 2 of the Logistic Growth Model 2

Solution: For the **discrete logistic growth model**

$$p_{n+1} = 2.7\, p_n - 0.0001\, p_n^2$$

the equilibria are found by substituting $p_e = p_n = p_{n+1}$

Thus,

$$
\begin{aligned}
p_e &= 2.7\, p_e - 0.0001\, p_e^2 \\
0 &= 1.7\, p_e - 0.0001\, p_e^2 = p_e(1.7 - 0.0001\, p_e)
\end{aligned}
$$

The equilibria satisfy

$$p_e = 0$$

and

$$1.7 - 0.0001\, p_e = 0 \quad \text{or} \quad p_e = 17,000$$

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Example 2 of the Logistic Growth Model   3

Solution (cont): For $f_2(p) = 2.7\,p - 0.0001\,p^2$,
the derivative satisfies

$$f_2'(p) = 2.7 - 0.0002\,p$$

At $p_e = 0$

$$f_2'(0) = 2.7 > 1$$

The solution monotonically grows away from this equilibrium,
as expected

At $p_e = 17,000$

$$f_2'(17,000) = 2.7 - 3.4 = -0.7$$

Since $-1 < f_2'(17,000) < 0$, the solution oscillates and
approaches this equilibrium
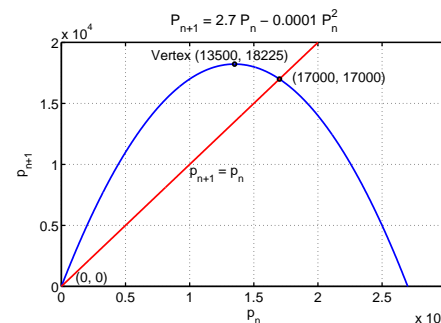This equilibrium is also **stable**

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Example 2 of the Logistic Growth Model   4

Graphing the updating function

- The p-intercepts are 0 and 27,000
- The vertex is at (13500, 18225)
- Below is graph of updating function and identity map with
 significant points

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Example 2 of the Logistic Growth Model   5

Simulation of

$$p_{n+1} = 2.7\,p_n - 0.0001\,p_n$$

with $p_0 = 100$ for 50 iterations
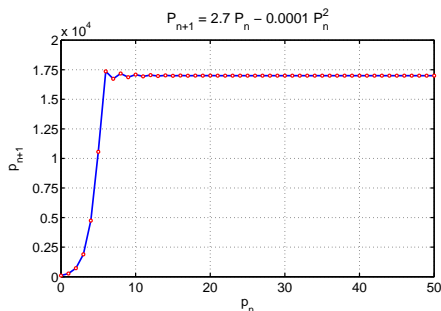
Simulation grows and overshoots the equilibrium, then oscillates
toward the equilibrium

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Example 3 of the Logistic Growth Model   1

Example 3: Consider the **discrete logistic growth model**

$$p_{n+1} = f_3(p_n) = 3.2\,p_n - 0.0001\,p_n^2$$

Skip Example

- Find all the equilibria for this model
- Determine the behavior of the solution near these equilibria
- Sketch a graph of the updating function and the identity
  map $p_{n+1} = p_n$
- Simulate the model, starting $p_0 = 100$ for 50 iterations

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Example 3 of the Logistic Growth Model 2

Solution: For the **discrete logistic growth model**

$$p_{n+1} = 3.2\,p_n - 0.0001\,p_n^2$$

the equilibria are found by substituting $p_e = p_n = p_{n+1}$

Thus,

$$
\begin{aligned}
p_e &= 3.2\,p_e - 0.0001\,p_e^2 \\
0 &= 2.2\,p_e - 0.0001\,p_e^2 = p_e(2.2 - 0.0001\,p_e)
\end{aligned}
$$

The equilibria satisfy

$$p_e = 0$$

and

$$2.2 - 0.0001\,p_e = 0 \quad \text{or} \quad p_e = 22,000$$

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Example 3 of the Logistic Growth Model 3

Solution (cont): For $f_3(p) = 3.2\,p - 0.0001\,p^2$, the derivative satisfies

$$f_3'(p) = 3.2 - 0.0002\,p$$

At $p_e = 0$

$$f_3'(0) = 3.2 > 1$$

The solution monotonically grows away from this equilibrium, as expected

At $p_e = 22,000$

$$f_3'(22,000) = 3.2 - 4.4 = -1.2 < -1$$

The solution oscillates away from this equilibrium
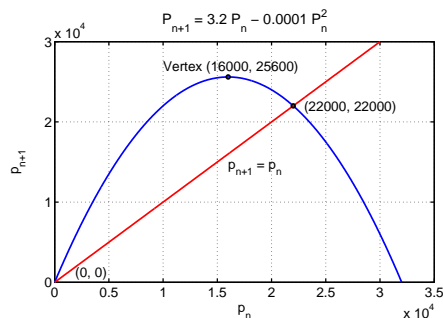This equilibrium is **unstable**

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Example 3 of the Logistic Growth Model 4

Graphing the updating function

- The p-intercepts are 0 and 32,000
- The vertex is at $(16000, 25600)$
- Below is graph of updating function and identity map with significant points
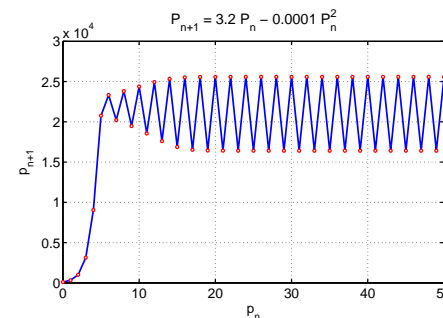

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Example 3 of the Logistic Growth Model 5

Simulation of

$$p_{n+1} = 3.2\,p_n - 0.0001\,p_n$$

with $p_0 = 100$ for 50 iterations

Simulation oscillates about the carrying capacity with period 2 behavior

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Example 4 - Logistic Growth with Emigration — 1

Logistic Growth with Emigration - Population growth may be affected by immigration or emigration

Skip Example

Consider the discrete dynamical population model

$$p_{n+1} = p_n + g(p_n) = 1.71\,p_n - 0.001\,p_n^2 - 7,$$

where n is measured in generations

- This model has a 71% growth rate per generation
- Logistic crowding effects are given by the term $0.001\,p_n^2$
- 7 individuals emigrate each generation

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Example 4 - Logistic Growth with Emigration — 2

Logistic Growth with Emigration

$$p_{n+1} = p_n + g(p_n) = 1.71\,p_n - 0.001\,p_n^2 - 7,$$

- Let $p_0 = 100$ and find the population for the next 3 generations
- Find the p-intercepts and the vertex for $g(p)$ and graph of $g(p)$
- By finding when the growth rate is zero, determine all equilibria for this model and analyze their stability

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Example 4 - Logistic Growth with Emigration — 3

Solution: We begin with $p_0 = 100$

$$
\begin{aligned}
p_1 &= p_0 + g(p_0) = 100 + 0.71(100) - 0.001(100)^2 - 7 = 154, \\
p_2 &= 154 + 0.71(154) - 0.001(154)^2 - 7 = 233, \\
p_3 &= 233 + 0.71(233) - 0.001(233)^2 - 7 = 337.
\end{aligned}
$$

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Example 4 - Logistic Growth with Emigration — 4

Solution (cont): The growth function satisfies

$$
\begin{aligned}
g(p) &= 0.71p - 0.001p^2 - 7 \\
g(p) &= -0.001(p^2 - 710p + 7000) \\
g(p) &= -0.001(p - 10)(p - 700)
\end{aligned}
$$

The p-intercepts are

$$p = 10 \qquad \text{or} \qquad p = 700$$
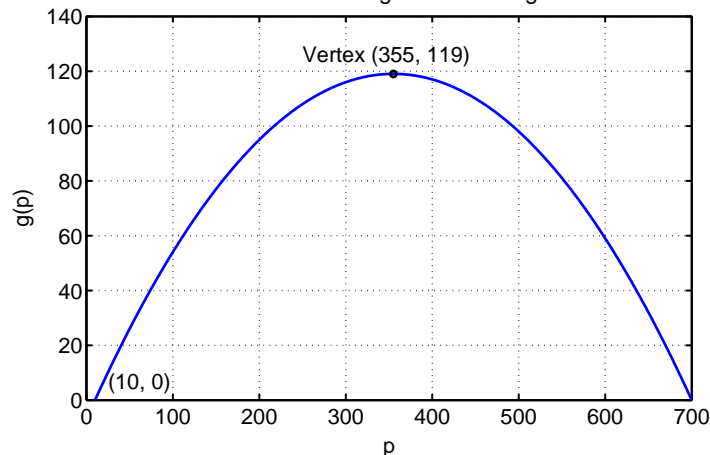
The vertex satisfies $p = 355$ with

$$g(355) = -0.001(345)(-345) = 119$$

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Example 4 - Logistic Growth with Emigration 5

Solution (cont): The graph of the growth function is

Growth Rate for Logistic with Emigration

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Example 4 - Logistic Growth with Emigration 6

Solution (cont): Equilibrium Analysis
Since the growth function $g(p)$ is zero at

$$p = 10 \qquad \text{and} \qquad p = 700,$$

these are the **equilibria**

The updating function is

$$F(p) = 1.71\,p - 0.001\,p^2 - 7$$

with derivative

$$F'(p) = 1.71 - 0.002\,p$$

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Example 4 - Logistic Growth with Emigration 7

Solution (cont): Stability Analysis With

$$F'(p) = 1.71 - 0.002\,p$$

At $p = 10$,

$$F'(10) = 1.69 > 1$$

so this equilibrium is **monotonically unstable (solutions growing away)**

At $p = 700$,

$$F'(700) = 0.31 < 1$$

so this equilibrium is **monotonically stable (solutions moving toward)**

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Example 5 - U. S. Census with 3 Growth Models 1

U. S. Census with Logistic Growth Model - This example uses the census data from 1790 to 2000 to compare 3 models

Skip Example

- Malthusian growth model

$$P_{n+1} = 1.1524\,P_n$$

- Nonautonomous growth model with n in decades after 1790

$$P_{n+1} = (1.3768 - 0.01473\,n)P_n$$

- Logistic growth model

$$P_{n+1} = f(P_n) = P_n + 0.2334\,P_n\left(1 - \frac{P_n}{411.1}\right)$$

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Example 5 - U. S. Census with 3 Growth Models 2

Malthusian growth model

$$P_{n+1} = (1 + r)P_n \quad \text{with} \quad P_0$$

- Least squares best fit to census data

$$P_n = P_0(1 + r)^n = 15.05(1.1524)^n$$

- The average growth over U. S. census history is $r = 0.1524$ per decade with best $P_0 = 15.05$ M
  - The sum of square errors is 2248
  - The P_0 is quite high and growth only matches growth near beginning of 20th century
- Malthusian model isn't expected to work well over long periods of time

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Example 5 - U. S. Census with 3 Growth Models 3

Nonautonomous growth model

$$P_{n+1} = (1 + k(t_n))P_n \quad \text{with} \quad P_0$$

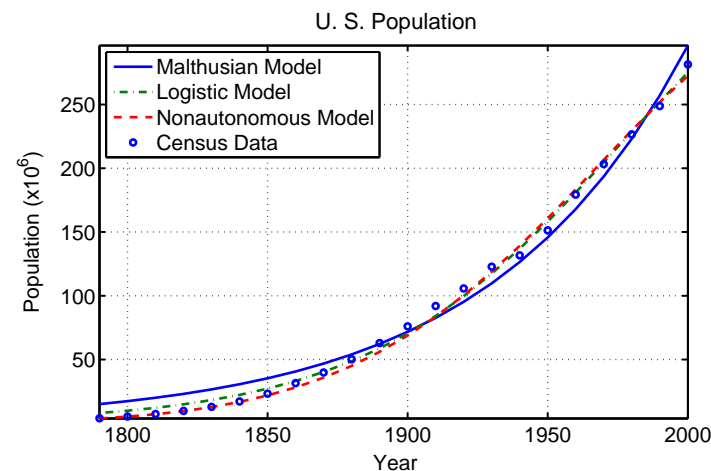- Best linear fit to growth over U. S. history is

$$k(t_n) = 0.3768 - 0.01473\,n$$

- Growth near 38% per decade early, declining about 1.5% per decade
- Least squares best fit to census data had $P_0 = 3.77$ M
  - The sum of square errors is 543
  - The P_0 is very close to actual 1790 census
- This model matches the census quite well, but model difficult to analyze mathematically

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Example 5 - U. S. Census with 3 Growth Models 4

Logistic growth model

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{M}\right) \quad \text{with} \quad P_0$$

- Least squares best fit to census data

$$P_{n+1} = P_n + 0.2334\,P_n\left(1 - \frac{P_n}{411.1}\right)$$

- This gives a growth rate of $r = 0.2334$ and carrying capacity of $M = 411.1$ with the best $P_0 = 8.04$
  - The sum of square errors is 479
  - The P_0 is high at 8.04 M
- This model matches the census data best of the 3 models

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Example 5 - U. S. Census with 3 Growth Models 5

Graph of the 3 models and U. S. census data

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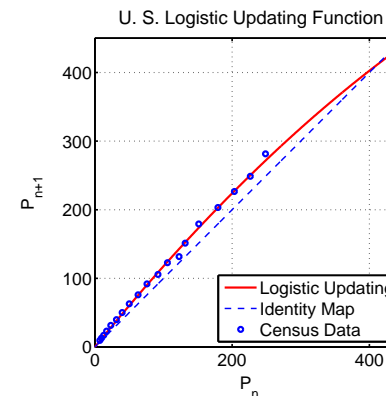## Example 5 - U. S. Census with 3 Growth Models   6

Logistic Updating Function

- Direct fitting of the logistic time series to data can be numerically unstable
- Finding the quadratic updating function uses stable numerical routines
- By plotting P_{n+1} versus P_n, one can see how the data compares to the updating function for the logistic growth model
- Find P_n and P_{n+1} by taking successive pairs of census data

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Graph of the Logistic Updating function

Graph shows U. S. census data, quadratic for logistic model, and identity map

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Example 5 - U. S. Census with 3 Growth Models   8

Logistic Updating function for U. S. census data

- The logistic updating function very closely follows the census data except at a couple of points
- The equilibria occur at the intersection of the updating function and the identity map
- The slope of the updating function at a point of intersection determines the stability of that equilibrium

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Example 5 - U. S. Census with 3 Growth Models   9

Logistic Updating function for U. S. census data

$$f(P_n) = P_n + 0.2334\, P_n \left(1 - \frac{P_n}{411.1}\right) = 1.2334\, P_n - 0.00056775\, P_n^2$$

The **equilibria** satisfy

$$
\begin{aligned}
P_e &= P_e + 0.2334\, P_e \left(1 - \frac{P_e}{411.1}\right) \\
0 &= 0.2334\, P_e \left(1 - \frac{P_e}{411.1}\right)
\end{aligned}
$$

The equilibria are

$$P_e = 0 \qquad \text{and} \qquad P_e = 411.1$$

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Updating Function

$$f(P) = 1.2334\,P - 0.00056775\,P^2$$

The **derivative of the updating function** is

$$f'(P) = 1.2334 - 0.0011355\,P$$

At the equilibrium, $P_e = 0$,

$$f'(0) = 1.2334 > 1$$

This equilibrium is **unstable** with solutions monotonically moving away

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Since the **derivative of the updating function** is

$$f'(P) = 1.2334 - 0.0011355\,P$$

At the equilibrium, $P_e = 411.1$,

$$f'(411.1) = 0.7666 < 1$$

This equilibrium is **stable** with solutions monotonically approaching the **carrying capacity**

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Example 5 - U. S. Census with 3 Growth Models 12

Summary: Future Projections

- The Malthusian growth model is simple but simulates poorly for the entire history of the U. S.
- Nonautonomous growth model
  - Simulates historical data well, but low by 3.2% in 2000 and 5.8% in 2010
  - Fails to account for recent immigration and high birth rates in immigrant community
  - Model predicts population increases to a maximum of 330 M around 2050, then declines
- Logistic growth model
  - Simulates historical data well, but low by 2.3% in 2000 and 4.0% in 2010 missing importance of recent immigration
  - Model predicts population increases to carrying capacity of 411.1 M, asymptotically

Cobwebbing 1

Consider the **discrete dynamical model**

$$p_{n+1} = f(p_n)$$

In the **Linear Discrete Dynamical Model section**, we showed a graphical method to view the local dynamics of this model called **cobwebbing**

Create a graph with the variable p_{n+1} on the vertical axis and p_n on the horizontal axis

Draw the graph of the **updating function**, $f(p_n)$ and the **identity map**

$$p_{n+1} = f(p_n) \qquad \text{and} \qquad p_{n+1} = p_n$$

Cobwebbing 2

Graphically, any intersection of the **updating function** and the **identity map**

$$p_{n+1} = f(p_n) \qquad \text{and} \qquad p_{n+1} = p_n$$

produces an equilibrium

- The process of **cobwebbing** shows the dynamics of this discrete dynamical model
- Start at some point p_0 on the horizontal axis, then go vertically to $f(p_0)$ to find p_1
- Next go horizontally to the line $p_{n+1} = p_n$
- Go vertically to $f(p_1)$ to find p_2
- The process is repeated to give a geometric interpretation of the dynamics of the discrete model

Cobwebbing – Breathing Model Example

Cobwebbing – Breathing Model Example

The model for a normal subject breathing an air mixture enriched with Ar satisfies the model

$$c_{n+1} = (1 - q)c_n + q\gamma = 0.82\,c_n + 0.0017$$

Below reviews the cobwebbing process for this example

Cobwebbing – Quadratic Example 1

Cobwebbing – Quadratic Example

- Breathing model has a simple **linear updating function**
  - Unique equilibrium
  - Monotonic dynamics
- **Quadratic updating function** allows complicated dynamics
  - Logistic growth model is a quadratic dynamical model
  - Have observed monotonic, oscillatory, and chaotic dynamics
  - Show oscillatory dynamics for

$$p_{n+1} = 3\,p_n(1 - p_n)$$

  using a few steps of cobwebbing
- This example has equilibria at 0 and $\frac{2}{3}$, the latter being between stable and unstable and oscillatory

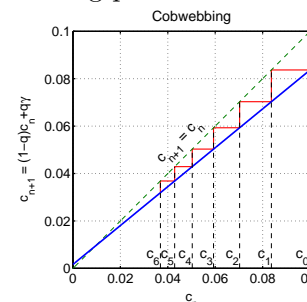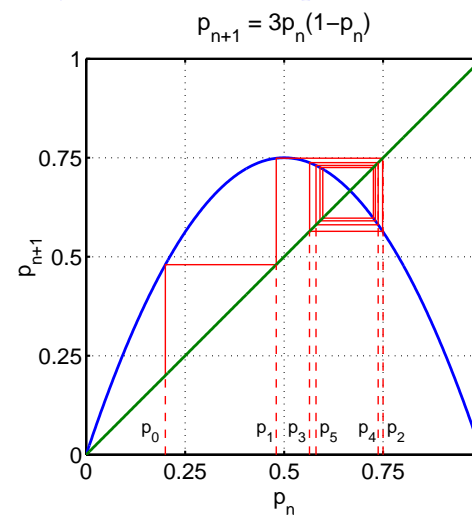## Cobwebbing – Quadratic Example 2

Cobwebbing – Quadratic Example