Calculus for the Life Sciences I

Lecture Notes – Linear Discrete Dynamical Models

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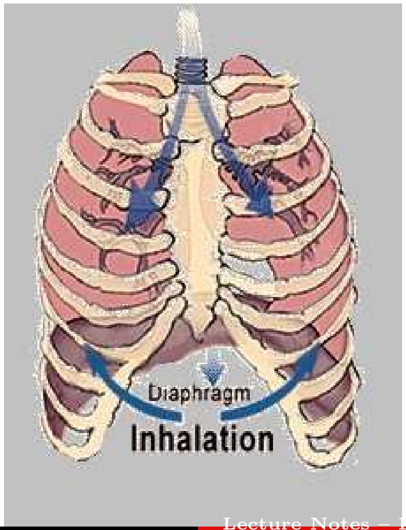
Outline

1. Lung Diseases and Modeling Breathing
2. Discrete Model for Breathing
3. Qualitative Analysis of Discrete Dynamical Models
   - Equilibria
   - Stability of a Linear Discrete Dynamical Model
   - General Stability
4. Malthusian Growth with Immigration or Emigration
5. Cobwebbing

Pulmonary Ventilation 1

Pulmonary Ventilation or Breathing



Pulmonary Ventilation 2

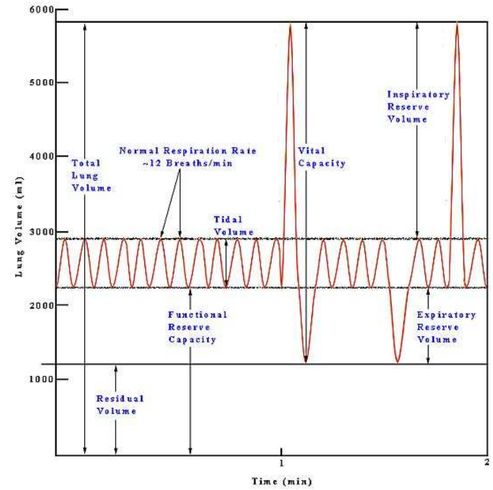
- **Pulmonary ventilation** or breathing brings oxygen to the cells of the body and removes metabolic waste product, carbon dioxide
- **Inspiration** or inflow of fresh air results from the contracting the muscles of the diaphragm
- Relaxation of these muscles or contraction of the abdominals causes **expiration** of air with the waste product CO_2
- Normal **respiration** in the lungs exchanges about 500 ml of air 12 times a minute
- **Tidal volume** is the normal volume of air inspired or expired
- The **inspiratory reserve volume** is about 3000 ml that can be inspired above the tidal volume

Pulmonary Ventilation 3

- The **expiratory reserve volume** is about 1100 ml, which can be forcefully expired
- The **vital capacity**, including all of the above, is about 4600 ml
- Well-trained athletes may have values **30-40% higher**
- Females have **20-25% less** for the quantities listed above
- The lungs contain surfactants, which prevent them from totally collapsing
  - Expelling all air requires too much energy to reinflate them from the collapsed state
  - Premature babies born before their genes for producing surfactants have turned on use more energy to breath than they derive from the process of breathing
- The **residual volume** is the amount of air that cannot be expelled and averages about 1200 ml

Pulmonary Ventilation 4

Pulmonary Ventilation or Breathing

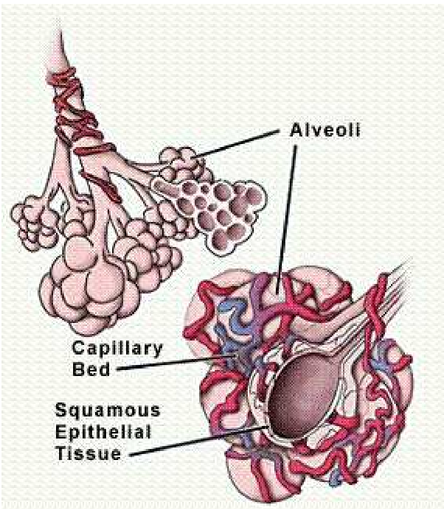


Pulmonary Ventilation 5

- The body needs oxygen for the tissues, which depends on respiration through the lungs
- Several respiratory diseases jeopardize the vital function of the lungs
  - The respiratory muscles damaged by spinal paralysis or poliomyelitis decreases the vital capacity to as low as 500 ml
  - The pulmonary compliance reduces vital capacity in diseases like tuberculosis, emphysema, chronic asthma, lung cancer, chronic bronchitis, cystic fibrosis, or fibrotic pleurisy
  - Several of the diseases above and heart disease can cause pulmonary edema, decreasing vital capacity from fluid build up
- The alveoli are where the oxygen actually enters the blood
- When the alveoli are damaged or filled with fluid (one result of smoking), the exchange of oxygen is inhibited

Pulmonary Ventilation 6

Alveoli and O_2 Exchange



Pulmonary Ventilation 7

- The vital capacity and residual volume help physiologists determine the health of the pulmonary system
- The vital capacity is easily measured by taking a deep breath and expiring into a spirometer
- Some of the diseases, like emphysema, are determined by the tidal volume and the functional residual capacity, the average or **minute respiratory volume**
- When the ratio of the tidal volume to the functional residual capacity becomes too low, then there is insufficient exchange of air to maintain adequate supplies of oxygen to the body

Discrete Model for Breathing 1

- Tidal volume and the functional residual capacity is found by breathing a mixture including an inert gas
  - The subject breathes the mixture until the lungs are essentially filled with this mixture
  - A physiologist measures the amount of the inert gas in a series of breaths after the subject is removed from the gas mixture to normal air
- The mathematical model for this experiment is a **discrete dynamical system**

Discrete Model for Breathing 2

- Professor Bruce Wingerd ran dilution experiments with the inert gas argon (Ar)
  - Argon is a noble gas, so is totally non-reactive
  - It is the third most common gas, comprising 0.93% of Earth's atmosphere (dry air has N_2 78%, O_2 21%, with CO_2 a distant fourth, 0.03%
- The subjects breathe an air mixture with 10% Ar
- Then the subjects resume breathing normal air at a normal rate

Discrete Model for Breathing 3

Normal Subject
Tidal Volume=550 ml

Breath Number	Percent Ar
0	0.100
1	0.084
2	0.070
3	0.059
4	0.050
5	0.043
6	0.037

Subject with Emphysema
Tidal Volume=250 ml

Breath Number	Percent Ar
0	0.100
1	0.088
2	0.078
3	0.069
4	0.061
5	0.055
6	0.049

Discrete Model for Breathing

Health of the Patient

- Vital capacity of the lungs is determined by a spirometer
- The experimental data are used to determine the **functional reserve volume** for the subjects
- Develop a model to analyze the data
- A physiologist uses this information to find the health of a subject's lungs

Discrete Model for Breathing

Breathing Experiment

- This breathing experiment is a dynamic exchange of gases, which occurs at discrete intervals of time
- Suggests a discrete dynamical model tracking the concentration of Ar in the lungs
- If c_n is the concentration of Ar at the end of the n^{th} inspiration cycle, we write a dynamical model

$$c_{n+1} = f(c_n)$$

Discrete Model for Breathing

Modeling Assumptions

- The concentration at the end of the $(n+1)^{st}$ inspiration cycle depends on remaining air in the lungs from the previous cycle and inhaling fresh air from the atmosphere
  - Assume the gases become well-mixed during this process, which ignores some of the complications caused by the actual physiological structures in the lungs, such as the "anatomical dead space" in the pharynx, trachea, and larger bronchi or weak mixing from slow gas flow in the alveoli
  - 500 ml of fresh air enters by inspiration, but only 350 ml reaches the alveoli, i.e., less than 1/7 exchange of gases with a normal breath

Discrete Model for Breathing

Physiological Parameters for Model

- V_i for the tidal volume (air normally inhaled and exhaled)
- V_r for the functional residual volume
- γ for the concentration of Ar in the atmosphere
- The fraction of atmospheric air exchanged in each breath is

$$q = \frac{V_i}{V_i + V_r}$$

- The residual air fraction is

$$1 - q = \frac{V_r}{V_i + V_r}$$

Discrete Model for Breathing

Model Development

- Upon exhaling, there remains behind the functional residual volume with the amount of Ar given by $V_r c_n$
- The inhaled air during this cycle contains the amount of Ar given by $V_i \gamma$
- The amount of Ar in the next breath is given by

$$V_r c_n + V_i \gamma$$

- The concentration of Ar is found by dividing by the total volume, $V_i + V_r$

Discrete Model for Breathing

Discrete Model for Breathing

- From definitions above,

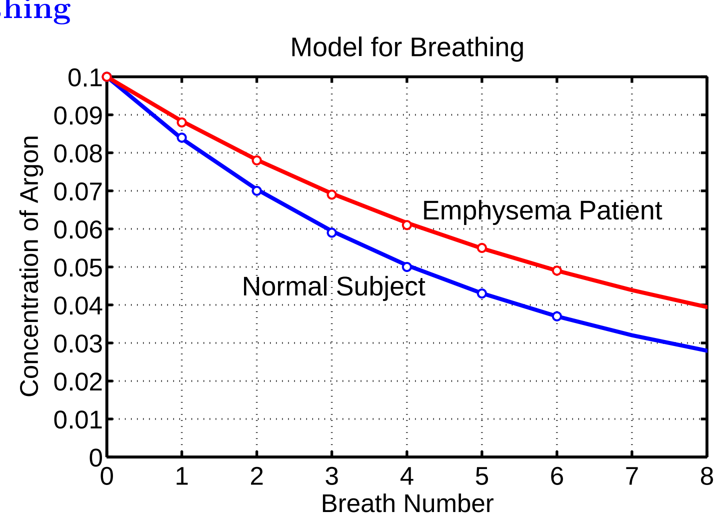
$$c_{n+1} = \frac{V_r c_n}{V_i + V_r} + \frac{V_i \gamma}{V_i + V_r}$$

- Use definitions of $q = \frac{V_i}{V_i + V_r}$ and $1 - q$
- **Linear Discrete Dynamical Model for Breathing an Inert Gas**

$$c_{n+1} = (1 - q)c_n + q\gamma$$

Discrete Model for Breathing

Graph of the data above with the best fitting model for breathing



Discrete Model for Breathing

Finding Functional Reserve Capacity

- The diseased states are often characterized by a decreased ratio between the tidal volume and the functional reserve capacity
- Emphysema is characterized by a loss of elasticity in the lungs and a decrease in the alveolar surface/volume ratio
- The discrete dynamical model for breathing an inert gas is solved for the parameter q

$$q = \frac{c_n - c_{n+1}}{c_n - \gamma}$$

Discrete Model for Breathing 12

Finding Functional Reserve Capacity

- Given tidal volume, V_i and q, the functional reserve capacity satisfies

$$V_r = \frac{1-q}{q} V_i$$

- **Normal Patient**
  - From data, $q = \frac{0.1-0.084}{0.1-0.0093} = 0.18$
  - With $V_i = 550$, then $V_r = \frac{0.82}{0.18} 550 = 2500$ ml
  - Ratio of tidal volume to functional reserve capacity is 0.22
- **Patient with Emphysema**
  - From data, $q = \frac{0.1-0.088}{0.1-0.0093} = 0.13$
  - With $V_i = 250$, then $V_r = \frac{0.87}{0.13} 250 = 1670$ ml
  - Ratio of tidal volume to functional reserve capacity is 0.15

Example – Lung Disease 1

Example: A subject with an unknown lung ailment enters the lab for testing. She is given a supply of air that has an enriched amount of argon gas (Ar). After breathing this supply of enriched gas, two successive breaths are measured with $c_1 = 0.0736$ and $c_2 = 0.0678$ of Ar. The model for breathing is given by

$$c_{n+1} = (1-q)c_n + q\gamma,$$

where $\gamma = 0.0093$.

- Find the fraction of air breathed, q
- What is the concentration of argon remaining in her lungs after 5 breaths?
- Assume that her tidal volume is measured to be $V_i = 220$. Find the functional reserve volume, V_r, where $q = \frac{V_i}{Vi+Vr}$

Skip Example

Example – Lung Disease 2

Solution: Since we are given γ and the two consecutive values c_1 and c_2, we can find q

$$\begin{aligned} c_2 &= (1-q)c_1 + q\gamma \\ 0.0678 &= (1-q)(0.0736) + 0.0093q \\ q &= 0.0902 \end{aligned}$$

The concentration after 5 breaths is found by simulation

$$\begin{aligned} c_2 &= 0.0678 \\ c_3 &= (1-0.0902)c_2 + 0.0093(0.0902) = 0.06252 \\ c_4 &= 0.9098(0.06252) + 0.000839 = 0.05772 \\ c_5 &= 0.9098(0.05772) + 0.000839 = 0.05336 \end{aligned}$$

Example – Lung Disease 3

Solution (cont): To find the functional reserve we recall

$$q = \frac{V_i}{V_i + V_r}$$

$$0.0902 = \frac{220}{220 + V_r}$$

This gives the functional reserve volume, $V_r = 2219$

This functional reserve is approximately normal.

The woman has a very low q, so the woman probably has some type of respiratory muscle disorder or neurological problem

Equilibria - Breathing Model

- The discrete dynamical model for breathing shows the concentration of Ar decreasing
- If the simulation for the normal individual is carried out for about 3 minutes or 36 breaths, it can be seen that the concentration of Ar drops to 0.0094, which is within 1% of the atmospheric concentration
- Since Ar is an inert gas, we expect that after breathing an enriched source of Ar, then the concentration should return to the same value as normally found in the atmosphere
- This value is the equilibrium value of Ar for the model

Equilibria

Equilibria for Discrete Dynamical Models

Consider a discrete dynamical system given by the equation

$$x_{n+1} = f(x_n)$$

where $f(x_n)$ is any function describing the dynamics of the model

An **equilibrium**, x_e, for this discrete dynamical system is achieved if $x_{n+1} = x_n = x_e$

The dynamic variable settles into a constant value for all n when

$$x_e = f(x_e)$$

Equilibrium - Breathing Model

Equilibrium for Discrete Breathing Model

Consider the **discrete breathing model**

$$c_{n+1} = (1-q)c_n + q\gamma$$

An **equilibrium**, c_e, for this model satisfies $c_{n+1} = c_n = c_e$ or

$$c_e = (1-q)c_e + q\gamma$$

Solving this

$$\begin{aligned} qc_e &= q\gamma \\ c_e &= \gamma \end{aligned}$$

Intuitively, this is the natural equilibrium, since eventually the concentration of an inert gas in the lungs will equalize to the atmospheric concentration

Linear Discrete Dynamical Model 1

General Linear Discrete Dynamical Model

$$y_{n+1} = a\,y_n + b$$

for constants a and b

The **equilibrium**, y_e, for this discrete dynamical model is

$$y_e = \frac{b}{1-a}$$

provided $a \neq 1$

The **equilibrium** is **positive** provided

- $a < 1$ and $b > 0$ or
- $a > 1$ and $b < 0$

Linear Discrete Dynamical Model 2

Since the **equilibrium** satisfies

$$y_e = \frac{b}{1-a}$$

The **equilibrium** is **negative** provided

- $a < 1$ and $b < 0$ or
- $a > 1$ and $b > 0$

Linear discrete dynamical models have a **single unique equilibrium** if the **slope** of the linear function, $a \neq 1$

If $a = 1$, then

- There are **no equilibria** when $b \neq 0$, or
- **All points are equilibria** when $b = 0$

Linear Discrete Dynamical Model 3

Stability of a Linear Discrete Dynamical Model

$$y_{n+1} = a\,y_n + b$$

An **equilibrium of a linear discrete dynamical model is stable** if either of the following conditions hold:

- Successive iterations of the model **approach the equilibrium**
- The slope $|a|$ is **less than 1**

An **equilibrium of a linear discrete dynamical model is unstable** if either of the following conditions hold:

- Successive iterations of the model **move away from the equilibrium**
- The slope $|a|$ is **greater than 1**

Example – Linear Discrete Dynamical Model 1

Example: Consider the **linear discrete dynamical model**

$$y_{n+1} = 1.05\,y_n - 200, \qquad y_0 = 2000$$

- Find the first 3 iterations, y_1, y_2, y_3
- Determine the equilibrium value
- Find the stability of the equilibrium

Skip Example

Example – Linear Discrete Dynamical Model 2

Solution: Let $y_0 = 2000$

$$\begin{aligned} y_1 &= 1.05\,y_0 - 200 = 1.05(2000) - 200 = 1900 \\ y_2 &= 1.05\,y_1 - 200 = 1.05(1900) - 200 = 1795 \\ y_3 &= 1.05\,y_2 - 200 = 1.05(1795) - 200 = 1684.75 \end{aligned}$$

The equilibrium satisfies

$$\begin{aligned} y_e &= 1.05\,y_e - 200 \\ 0.05\,y_e &= 200 \\ y_e &= 4000 \end{aligned}$$

The **equilibrium is unstable**, since iterations are moving away from the equilibrium and the slope $a = 1.05 > 1$.

Example – Linear Discrete Dynamical Model 1

Example: Consider the **linear discrete dynamical model**

$$y_{n+1} = 0.6\, y_n + 50, \qquad y_0 = 100$$

- Find the first 3 iterations, y_1, y_2, y_3
- Determine the equilibrium value
- Find the stability of the equilibrium

Skip Example

Example – Linear Discrete Dynamical Model 2

Solution: Let $y_0 = 100$

$$\begin{aligned} y_1 &= 0.6\, y_0 + 50 = 0.6(100) + 50 = 110 \\ y_2 &= 0.6\, y_1 + 50 = 0.6(110) + 50 = 116 \\ y_3 &= 0.6\, y_2 + 50 = 0.6(116) + 50 = 119.6 \end{aligned}$$

The equilibrium satisfies

$$\begin{aligned} y_e &= 0.6\, y_e + 50 \\ 0.4\, y_e &= 50 \\ y_e &= 125 \end{aligned}$$

The **equilibrium is stable**, since iterations are moving towards the equilibrium and the slope $a = 0.6 < 1$.

Stability of Discrete Dynamical Model

A discrete dynamical model is given by the equation

$$x_{n+1} = f(x_n)$$

for some updating function, $f(x_n)$. Suppose that this model has an equilibrium, x_e, so $x_e = f(x_e)$.

The equilibrium, x_e, is said to be **stable** if one starts with some x_0 "near" the equilibrium, then subsequent iterations of the model have x_n approaching the equilibrium.

The equilibrium, x_e, is said to be **unstable** if one starts with some x_0 "near" the equilibrium, then subsequent iterations of the model have x_n moving away from the equilibrium.

Malthusian Growth with Immigration or Emigration 1

Population Growth Models

- The discrete Malthusian growth model worked for the U. S. population over only a limited time
- Malthusian model doesn't account for time-varying effects, resource limitation, immigration, etc.
- Time-varying model substantially improved, but difficult to analyze
- Through much of the 20^{th} century, the government has regulated legal immigration to $250,000$ people per year
- Models with immigration or emigration are not **closed** systems

Malthusian Growth with Emigration 2

Malthusian Growth with Emigration

$$P_{n+1} = (1+r)P_n - \mu$$

where r is a rate of growth and μ is a constant population emigrating in each time interval

Given an initial population, P_0

$$\begin{aligned} P_1 &= (1+r)P_0 - \mu, \\ P_2 &= (1+r)P_1 - \mu = (1+r)((1+r)P_0 - \mu) - \mu, \\ &= (1+r)^2 P_0 - ((1+r)+1)\mu \end{aligned}$$

Malthusian Growth with Emigration 3

Continuing the iterations

$$\begin{aligned} P_3 &= (1+r)^3 P_0 - ((1+r)^2 + (1+r) + 1)\mu, \\ &\vdots \\ P_n &= (1+r)^n P_0 - ((1+r)^{n-1} + \ldots + (1+r) + 1)\mu, \end{aligned}$$

which simplifies to

$$P_n = (1+r)^n P_0 - \frac{((1+r)^n - 1)}{r}\mu.$$

Though complicated, this is a **closed form solution** depending only upon P_0, r, n, and μ

Example – Malthusian Growth with Immigration 1

Example of Malthusian Growth with Immigration: Assume that a population of animals in a lake satisfies

$$P_{n+1} = (1+r)P_n + \mu$$

where r is a rate of growth and μ is a constant population immigrating to the lake in each time interval

Assume a census for 3 successive weeks gives populations, $P_0 = 500$, $P_1 = 670$, and $P_2 = 874$

Find the rate of growth r and immigration rate μ, then determine the populations expected in the next two weeks

Example – Malthusian Growth with Immigration 2

Solution: From the data with the model

$$\begin{array}{ccc} P_1 = (1+r)P_0 + \mu & \text{and} & P_2 = (1+r)P_1 + \mu \\ 670 = (1+r)500 + \mu & \text{and} & 874 = (1+r)670 + \mu \end{array}$$

Subtract the first equation from the second equation

$$\begin{aligned} 204 &= (1+r)(670 - 500) \\ 1 + r &= \frac{204}{170} = 1.2 \\ r &= 0.2 \end{aligned}$$

From the model

$$\begin{aligned} 670 &= 1.2(500) + \mu \\ \mu &= 670 - 600 = 70 \end{aligned}$$

Example – Malthusian Growth with Immigration 3

Solution (cont): The model is given by

$$P_{n+1} = 1.2\,P_n + 70$$

Since $P_2 = 874$,

$$\begin{aligned} P_3 &= 1.2\,P_2 + 70 = 1.2(874) + 70 = 1118.8 \\ P_4 &= 1.2(1118.8) + 70 = 1412.56 \end{aligned}$$

This model has no positive equilibrium, and it grows unbounded (almost exponentially)

Solution of Linear Discrete Model

Consider the **linear discrete model**

$$y_{n+1} = a\,y_n + b$$

The general solution is given by

$$y_n = a^n y_0 + \frac{(a^n - 1)}{(a - 1)} b$$

$(a \neq 1)$

Solution depends on parameters a and b, initial condition y_0, and time, n

Cobwebbing 1

Consider the **discrete dynamical model**

$$x_{n+1} = f(x_n)$$

There is an easy way to graphically view the local dynamics of this model

The state of the model x_{n+1} is a function of the state x_n

Create a graph with the variable x_{n+1} on the vertical axis and x_n on the horizontal axis

Draw the graph of the **updating function**, $f(x_n)$ and the **identity map**

$$x_{n+1} = f(x_n) \qquad \text{and} \qquad x_{n+1} = x_n$$

Cobwebbing 2

Graphically, any intersection of the **updating function** and the **identity map**

$$x_{n+1} = f(x_n) \qquad \text{and} \qquad x_{n+1} = x_n$$

produces an equilibrium

- The process of **cobwebbing** shows the dynamics of this discrete dynamical model
- Start at some point x_0 on the horizontal axis, then go vertically to $f(x_0)$ to find x_1
- Next go horizontally to the line $x_{n+1} = x_n$
- Go vertically to $f(x_1)$ to find x_2
- The process is repeated to give a geometric interpretation of the dynamics of the discrete model

Cobwebbing – Breathing Model Example 1

Cobwebbing – Breathing Model Example

The model for a normal subject breathing an air mixture enriched with Ar satisfies the model

$$c_{n+1} = (1-q)c_n + q\gamma = 0.82\, c_n + 0.0017$$

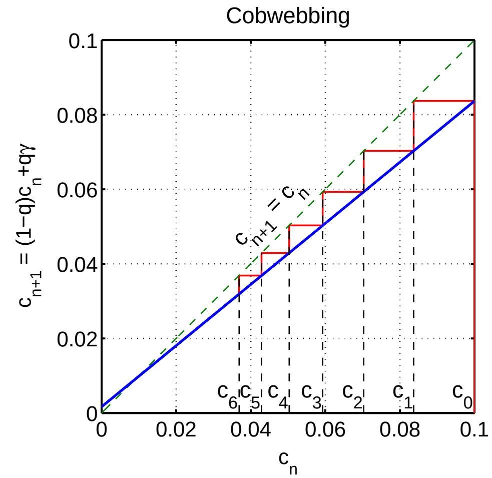
This model has a **stable** equilibrium ($c_e = \gamma = 0.0093$)

Initially, there was 10% Ar in the mixture, so $c_0 = 0.1$, and subsequent breaths had the concentration of Ar tending to γ

Cobwebbing – Breathing Model Example 2

Cobwebbing – Breathing Model Example

Below shows the cobwebbing process for this example



Cobwebbing – Population Model with Emigration 1

Cobwebbing – Population Model with Emigration

The model for a population with Emigration satisfies the model

$$P_{n+1} = (1+r)P_n - \mu = 1.2\, P_n - 500$$

This model has an **unstable** equilibrium ($P_e = \frac{\mu}{r} = 2500$)

- If the population begins below 2500, then the population goes to extinction
- If the population begins above 2500, then the population grows almost exponentially

Cobwebbing –Population Model with Emigration 2

Cobwebbing – Population Model with Emigration

Below shows the cobwebbing process for this example