Calculus for the Life Sciences I

Lecture Notes – Limits, Continuity, and the Derivative

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Outline

Introduction

- Limits are central to Calculus

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- Present definitions of limits, continuity, and derivative

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- Graphically show these ideas
- Recall derivative is related to the slope of the tangent line
- Complete understanding of the definitions is beyond the scope of this course

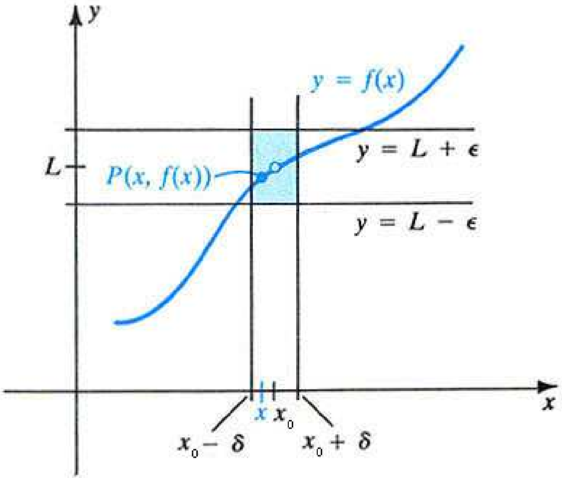
Definition of Limit

Limits – Conceptually, the **limit of a function** $f(x)$ at some point x_0 simply means that if your value of x is very close to the value x_0, then the function $f(x)$ stays very close to some particular value

Definition: The **limit of a function** $f(x)$ at some point x_0 exists and is equal to L if and only if every "small" interval about the limit L, say the interval $(L-\epsilon, L+\epsilon)$, means you can find a "small" interval about x_0, say the interval $(x_0-\delta, x_0+\delta)$, which has all values of $f(x)$ existing in the former "small" interval about the limit L, except possibly at x_0 itself

Definition of Limit

Diagram for Definition of Limit



Examples of Limits

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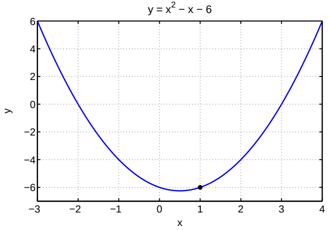

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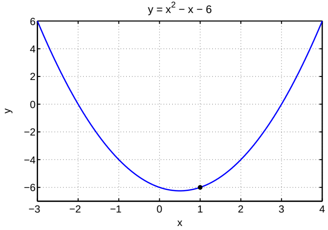

Fact: Any **polynomial**, $p(x)$, has as its limit at some x_0, the value of $p(x_0)$

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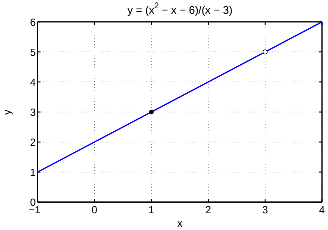

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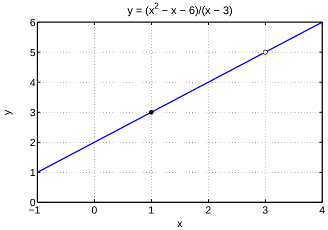

Fact: Any **rational function**, $r(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with $q(x_0)$ not zero, then the limit exists with the limit being $r(x_0)$

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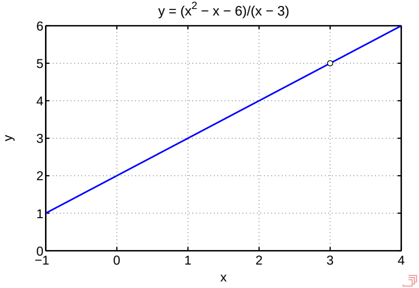
- Its limit exists though the function is not defined at $x_0 = 3$



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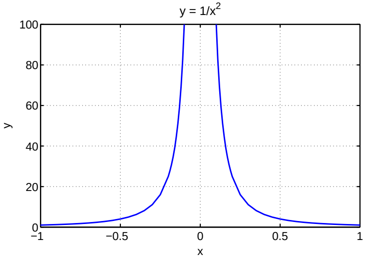

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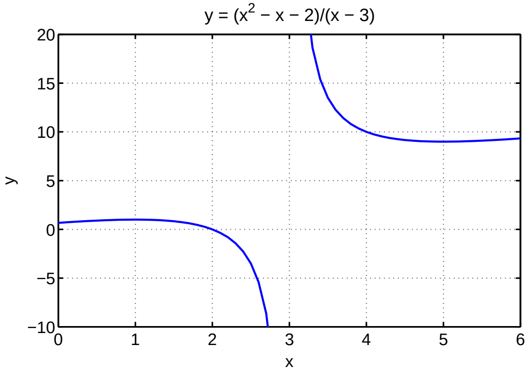

Fact: Whenever you have a **vertical asymptote** at some x_0, then the **limit fails to exist** at that point

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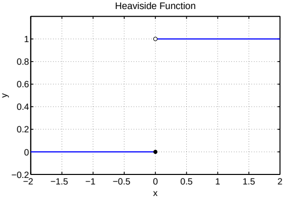

Perspective: Whenever a **function is defined differently on different intervals** (like the Heaviside function), check the x-values where the function changes in definition to see if the function has a limit at these x-values

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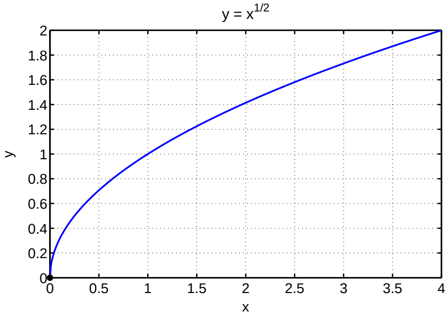

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  - Special cases like the square root function

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Definition: A function $f(x)$ is **continuous** at a point x_0 if the limit exists at x_0 and is equal to $f(x_0)$

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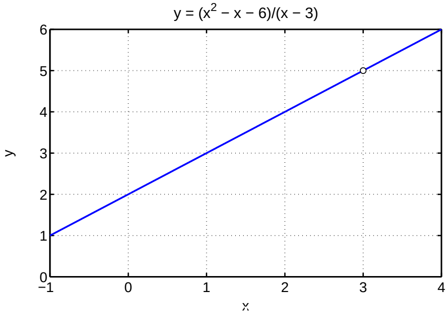

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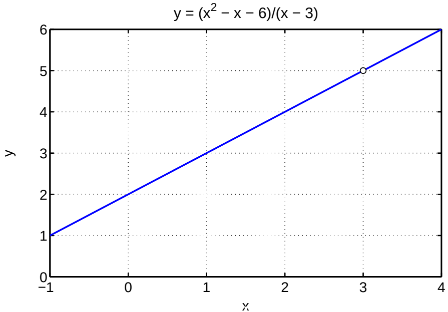

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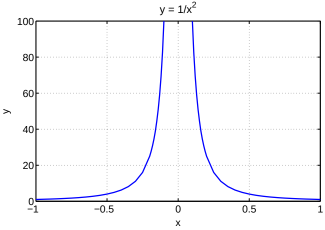

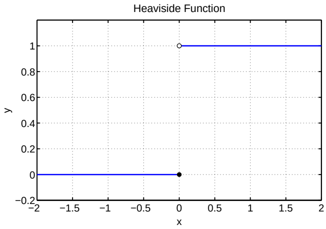

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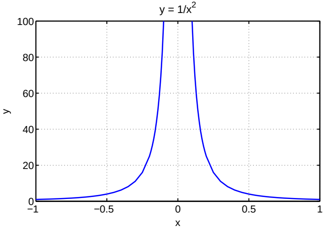

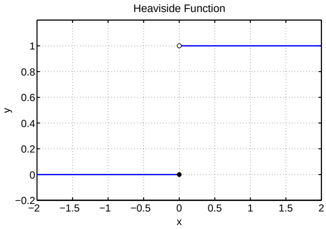

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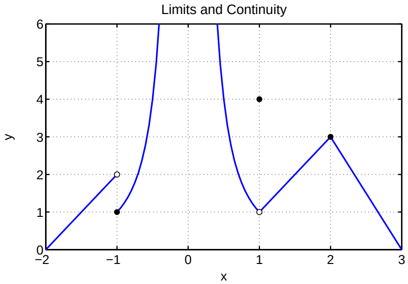

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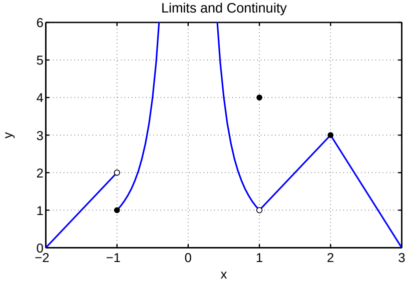

Difficulties with this function occur at integer values

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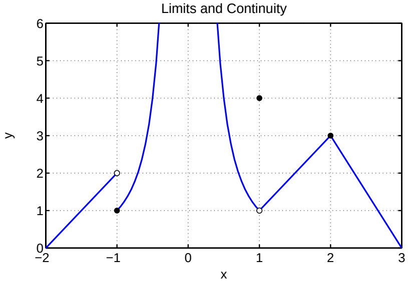

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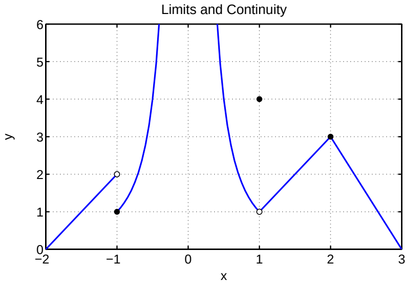

At $x = -1$, the function has the value $f(-1) = 1$

The function is not continuous nor does a limit exist at this point

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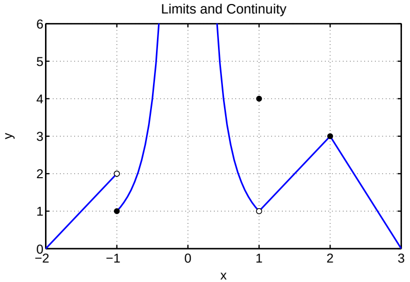

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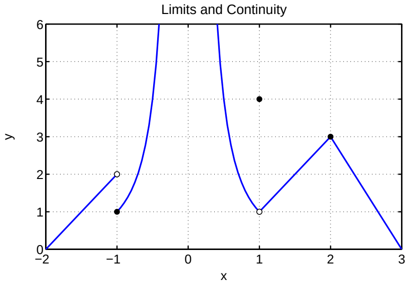

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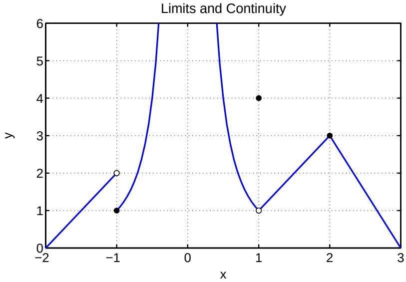

At $x = 1$, the function has the value $f(1) = 4$

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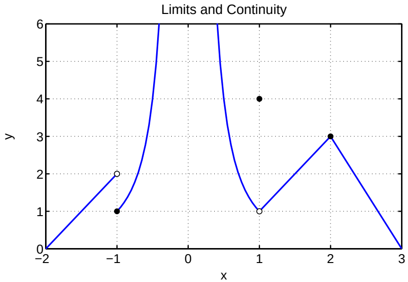

At $x = 1$, the function has the value $f(1) = 4$

The function is not continuous, but the limit exists with

$$\lim_{x \to 1} f(x) = 1$$

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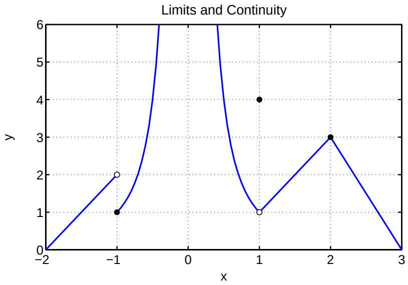

At $x = 2$, the function is continuous with $f(2) = 3$, which also means that the limit exists

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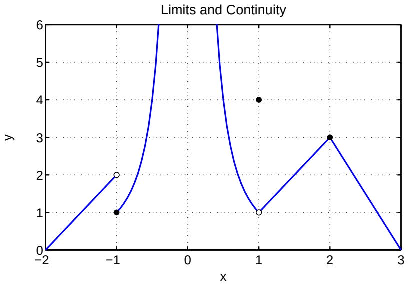
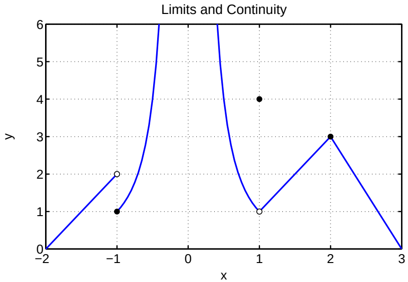


At all non-integer values of x the function is continuous (hence its limit exists)

Comparing Limits and Continuity



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We will see that the derivative only exists at these non-integer values of x

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Definition: The **derivative of a function** $f(x)$ at a point x_0 is denoted $f'(x_0)$ and satisfies

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists

Derivative of x^2

Example: Use the definition to find the derivative of

$$f(x) = x^2$$

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&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} (2x + h) \\
&= 2x
\end{aligned}$$

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$$f(x) = \frac{1}{x+2}, \qquad x \neq -2$$

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- In Lab, a very easy way to find derivatives is using the Maple **diff** command