Calculus for the Life Sciences I

Lecture Notes – Introduction to the Derivative

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Spring 2013

Outline

Introduction to the Derivative

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- The derivative is very important to Calculus

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- How to view the Derivative

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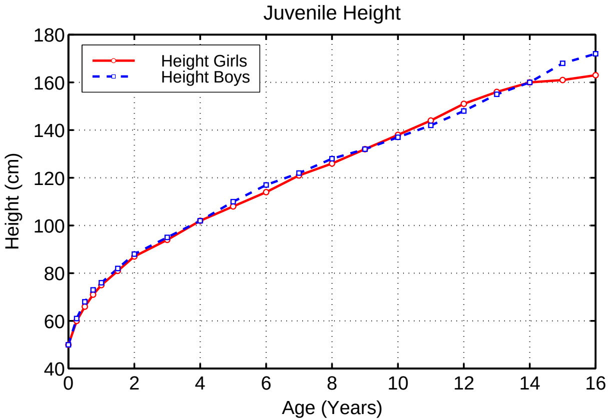
- The derivative is very important to Calculus
- How to view the Derivative
  - Rate of Growth
  - Velocity
  - Geometric view: The Tangent Line

Juvenile Height

Graph of the heights of girls and boys ages 0 to 18



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  - The later years show the growth rate slowing
- Growth rate is the difference in heights divided by the difference in time measured in years
- The growth rate $g(t)$ is approximated by the formula

$$g(t_0) = \frac{h(t_1) - h(t_0)}{t_1 - t_0}$$

where t_0 and t_1 are successive ages with heights $h(t_0)$ and $h(t_1)$

Juvenile Height

Growth Rate for Children

Girls age from 2 to 3

Age (years)	Height (cm)	Annual Growth Rate (cm/yr)
$t_0 = 2$	$h(t_0)$=87	
$t_1 = 3$	$h(t_1) = 94$	$g(2) = \dfrac{h(3) - h(2)}{3 - 2} = 7$

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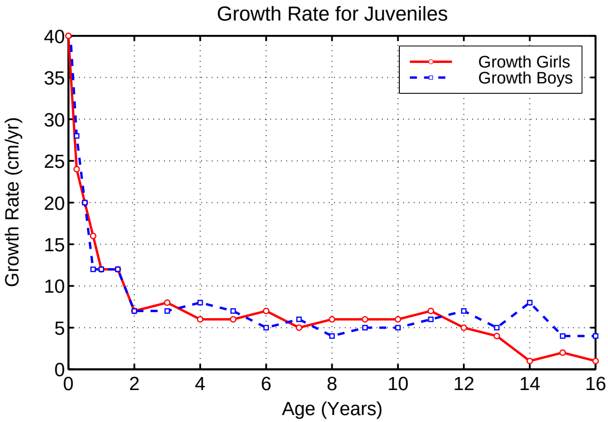
Boys age 3 months to 6 months

Age (years)	Height (cm)	Annual Growth Rate (cm/yr)
$t_0 = 0.25$ (3 months)	$h(t_0)$=61	
$t_1 = 0.5$ (6 months)	$h(t_1) = 68$	$g(0.25) = \dfrac{h(0.5) - h(0.25)}{0.5 - 0.25} = 28$

Juvenile Height

Growth Rate for Children



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Juvenile Height

Growth Rate for Children

- The growth rate is higher for early years
- Stays almost constant for many years
- Drops almost to zero in the late teens

Yeast Population Growth 1

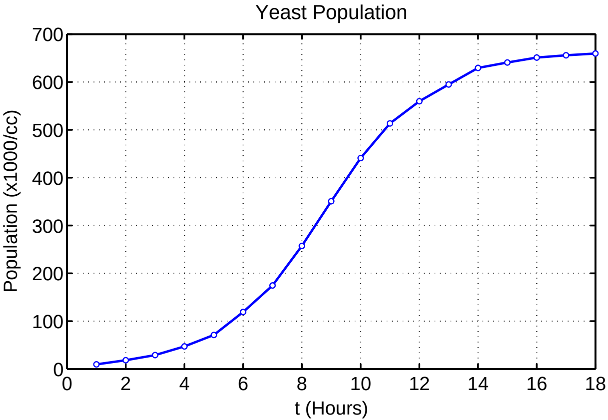
Growing Culture of Yeast - Carlson (1913)

Population in thousands/cc

Time	Population	Time	Population	Time	Population
1	9.6	7	174.6	13	594.8
2	18.3	8	257.3	14	629.4
3	29.0	9	350.7	15	640.8
4	47.2	10	441.0	16	651.1
5	71.1	11	513.3	17	655.9
6	119.1	12	559.7	18	659.6

Yeast Population Growth

Population of Yeast



Yeast Population Growth 3

Graph of Population of Yeast

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  - This shape occurs frequently in biologically models
- The growth rate for the population of yeast
  - Slow for early hours
  - Increases to a maximum near 8 hours
  - Decreases and levels off as the population reaches its carrying capacity

Yeast Population Growth

Growth of the Population of Yeast

Define the population at each hour as $P(t)$

Yeast Population Growth

Growth of the Population of Yeast

Define the population at each hour as $P(t)$

The growth of the yeast for each hour is computed

$$g(t_n) = \frac{P(t_{n+1}) - P(t_n)}{t_{n+1} - t_n}$$

Yeast Population Growth

Growth of the Population of Yeast

Define the population at each hour as $P(t)$

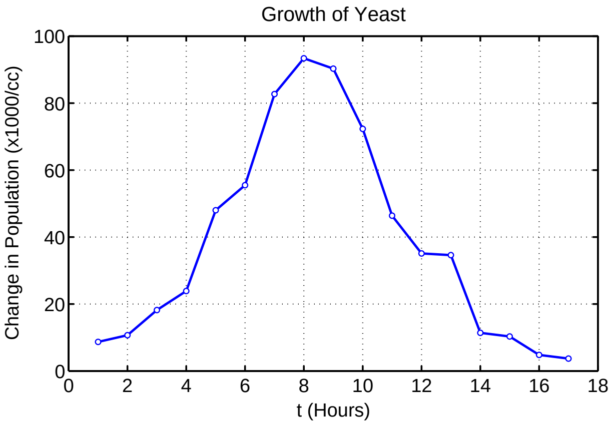
The growth of the yeast for each hour is computed

$$g(t_n) = \frac{P(t_{n+1}) - P(t_n)}{t_{n+1} - t_n}$$

This is the slope of the curve computed between each of the data points

Yeast Population Growth

Growth of the Population of Yeast



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- The growth graph shows the early slow growth, peaking at 8 hours, then decreasing toward zero
- With more data, the growth curve would be smoother
- It is **very important** to note that the graph of the yeast population and the graph of the growth of the yeast population are different graphs, but are related through the **slope of the population graph**
- The **Derivative** will be the **instantaneous growth rate at any time** for any population curve

Example – Growth of a Puppy

Developing his Project Calculus course, David Smith measured the growth of his Golden Retriever puppy, Sassafras

Age (days)	Weight (lbs)	Age (days)	Weight (lbs)
0	3.25	101	30
10	4.25	115	37
20	5.5	150	54
30	7	195	65
40	9	230	70
50	11.5	332	75
60	15	436	77
70	19		

Skip Example

Example – Growth of a Puppy 2

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$$g_{ave} = \frac{19 - 3.25}{10} = 1.575 \text{ lb/week}$$

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Estimate the weekly growth rate of the puppy at age 10 weeks using the data at 70 and 101 days

Solution: The weekly growth rate at 10 weeks satisfies

$$g_{ave} = \frac{30 - 19}{(101 - 70)/7} = 2.48 \text{ lb/week}$$

Example – Growth of a Puppy

What is the weekly growth rate between days 230 and 436?

Example – Growth of a Puppy 3

What is the weekly growth rate between days 230 and 436?

Solution: The weekly growth rate between 230 and 436 days satisfies

$$\frac{77-70}{(436-230)/7} = 0.238 \text{ lb/week}$$

Example – Growth of a Puppy 3

What is the weekly growth rate between days 230 and 436?

Solution: The weekly growth rate between 230 and 436 days satisfies

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The growth rate increases for several weeks, then slows down as the puppy matures

Example – Growth of a Puppy 3

What is the weekly growth rate between days 230 and 436?

Solution: The weekly growth rate between 230 and 436 days satisfies

$$\frac{77-70}{(436-230)/7} = 0.238 \text{ lb/week}$$

The growth rate increases for several weeks, then slows down as the puppy matures

Once again, the growth curve for the weight of a puppy gives a typical **S-shaped curve**

Derivative as a Velocity

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- Differential Calculus – developed in the 17^{th} century by Sir Isaac Newton and Gottfried Leibnitz to explain the physics of motion
- Velocity of an object is the change in distance divided by the change in time
  - If we travel 200 ft in 10 sec, then we had an average velocity of 20 ft/sec

Trotting Horses 1

Eadweard Muybridge – Trotting Horse

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Eadweard Muybridge – Trotting Horse

- In the 1800s, there was a controversy whether or not a trotting horse ever had all feet off of the ground
- Photographer Eadweard Muybridge developed some special photographic techniques for viewing animals and humans in motion by collecting timed sequences of still pictures
- Viewed in succession with the same intervening times, these pictures produce an animation of motion, which was a precursor to modern motion pictures

Trotting Horses

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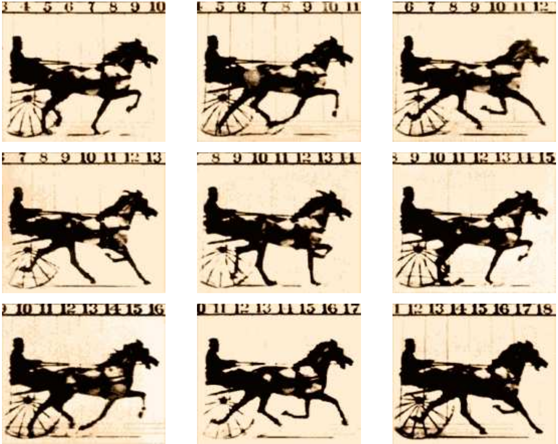

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- How do you determine the speed of a cheetah or the velocity of a peregrine falcon?
- Above sequence of pictures has units of distance (ft) marked in the background and units of time on each frame
- Start by choosing a reference point, say the man's head

Trotting Horses

Trotting Horse Speed

The position at $t_0 = 0$ sec, satisfies $s(t_0) = 3.5$ ft, while at $t_1 = 0.04$ sec, the head is at $s(t_1) = 4.5$ ft

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At $t_2 = 0.08$ sec, the head is at $s(t_2) = 5.6$ ft, so the velocity satisfies

$$v(t_1) = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{5.6 - 4.5}{0.08 - 0.04} = 27.5 \text{ ft/sec} \ = 18.75 \text{ mph}$$

Trotting Horses 4

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This is approximately the same

Trotting Horses 5

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$$v(t_{ave}) = \frac{s(t_f) - s(t_0)}{t_f - t_0} = \frac{11.5 - 3.5}{0.32 - 0} = 25.0 \text{ ft/sec} \ = 17.05 \text{ mph}$$

Thus, the velocity is relatively constant over the short time interval of the pictures

Trotting Horses

More details

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- How would you answer this question?
- Probably want many more pictures at smaller time intervals
- **This is the limiting process we will undertake to find the derivative**

Example –Ball Falling under the Influence of Gravity 1

A Ball Falling under the Influence of Gravity

A steel ball is dropped from a height of 4 meters and has its height measured every 0.1 seconds

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Time (sec)	Distance (cm)	Time (sec)	Distance (cm)
0	0	0.5	123
0.1	5	0.6	176
0.2	19	0.7	240
0.3	44	0.8	313
0.4	78	0.9	396

Falling under the Influence of Gravity 2

Find the average speed of the ball over the 0.9 seconds of the experiment

Falling under the Influence of Gravity 2

Find the average speed of the ball over the 0.9 seconds of the experiment

Solution:

$$v_{ave} = \frac{396 - 0}{0.9 - 0} = 440.0 \text{ cm/sec}$$

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Determine the average speed of the ball between 0.5 and 0.7 seconds

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$$v_{ave} = \frac{396 - 0}{0.9 - 0} = 440.0 \text{ cm/sec}$$

Determine the average speed of the ball between 0.5 and 0.7 seconds

Solution:

$$v_{ave} = \frac{240 - 123}{0.7 - 0.5} = 585 \text{ cm/sec}$$

Example – Sky Diving 1

A sky diver encounters a significant amount of air resistance when free falling (and more significantly when the parachute opens), so his speed will not match the parabolic curve characteristic of the Falling Ball

Time (sec)	Height (ft)	Time (sec)	Height (ft)
0	10,000	25	5,763
5	9,633	30	4,733
10	8,797	35	3,703
15	7,811	40	2,673
20	6,791	45	1,643

Skip Example

Example – Sky Diving 2

- Graph the Height *vs.* Time

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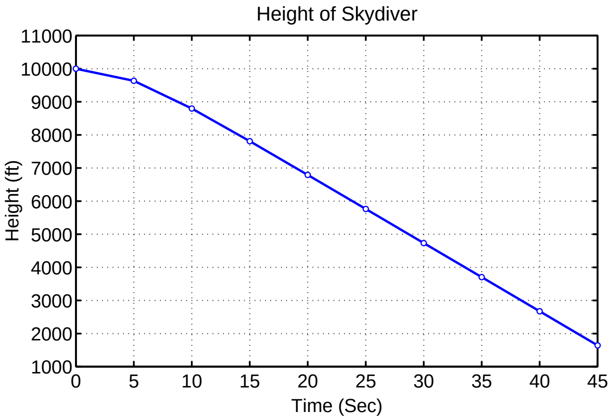
Example – Sky Diving 2

- Graph the Height *vs.* Time
- Compute the approximate velocity using the successive rows of the table
- Graph the velocity curve
- What is the approximate velocity in miles per hour at 30 seconds into the fall?
- Can you estimate when the sky diver would hit the ground if the parachute failed to open?

Example – Sky Diving

Solution: Graph the Height *vs.* Time



Example – Sky Diving

Solution: Compute the first two velocity points of the graph:

$$
\begin{aligned}
v(t_0) &= \frac{h(t_1) - h(t_0)}{t_1 - t_0} = \frac{9,633 - 10,000}{5 - 0} = -73.4 \text{ ft/sec}, \\
v(t_1) &= \frac{h(t_2) - h(t_1)}{t_2 - t_1} = \frac{8,797 - 9,633}{10 - 5} = -167.2 \text{ ft/sec}.
\end{aligned}
$$

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Solution: Compute the first two velocity points of the graph:

$$v(t_0) = \frac{h(t_1) - h(t_0)}{t_1 - t_0} = \frac{9,633 - 10,000}{5 - 0} = -73.4 \text{ ft/sec},$$

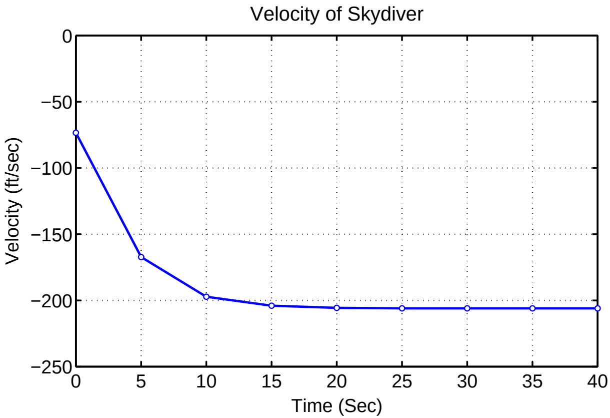
$$v(t_1) = \frac{h(t_2) - h(t_1)}{t_2 - t_1} = \frac{8,797 - 9,633}{10 - 5} = -167.2 \text{ ft/sec}.$$

The velocity at 30 seconds is

$$v(t_6) = \frac{3703 - 4733}{35 - 30} = -206 \text{ ft/sec},$$

Example – Sky Diving

Solution: Graph the velocity curve using calculations above



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At 45 sec, the sky diver is at 1643 ft and traveling at v_{term}

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We get, time = distance/v = 1643/206 = 8.0 sec to cover the remaining 1643 ft

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The **terminal velocity** is approximately the velocity at 30 sec or $v_{term} = -206$ ft/sec (which is about -140.5 mph)

At 45 sec, the sky diver is at 1643 ft and traveling at v_{term}

We get, time = distance/v = $1643/206 = 8.0$ sec to cover the remaining 1643 ft

Hence, the sky diver would fall for about $45 + 8 = 53$ sec if the parachute failed