Calculus for the Life Sciences I

Lecture Notes – Function Review

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Outline

Rate of mRNA Synthesis

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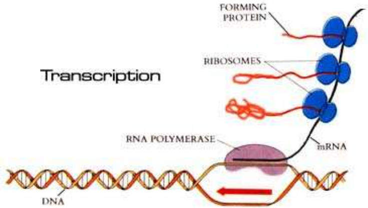
- DNA code used either for all aspects of the growth, maintenance, and reproduction of the cell
- The synthesis of proteins follows the processes of transcription and translation
- Proteins key for all cellular processes

Transcription

Transcription of a bacterial gene



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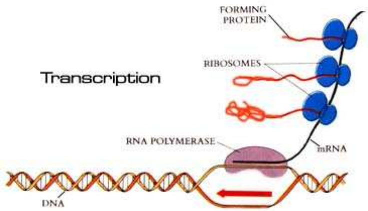
- A controlled sequence of steps, RNA polymerase, reads genetic code and produces a complementary messenger RNA (mRNA) template



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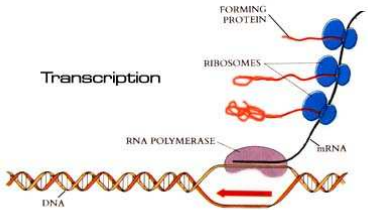
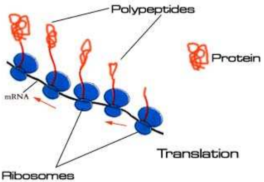
- A controlled sequence of steps, RNA polymerase, reads genetic code and produces a complementary messenger RNA (mRNA) template
- The mRNA is a short-lived blueprint for the production of a specific protein with a particular activity



Translation

Translation of a bacterial mRNA



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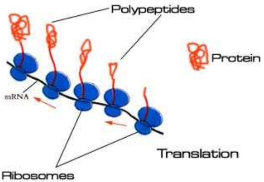
- Begins shortly after transcription starts, with ribosomes reading the triplet codons on the mRNA



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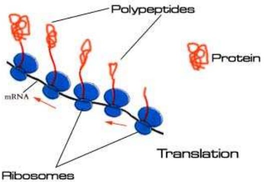

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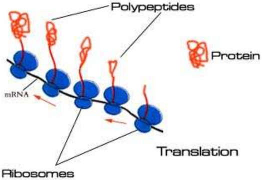
- Begins shortly after transcription starts, with ribosomes reading the triplet codons on the mRNA
- Ribosome assembles a series of specific amino acids, forming a polypeptide
- Polypeptide probably folds passively into a tertiary structure which often combines with other proteins to become active or an enzyme



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- The rate of growth of a bacterial cell depends on the rate at which it assembles all of its cellular components inside the cell
- The rate of production of different components inside the cell varies depending on the length of time it takes for a cell to double
- The table below shows the doublings/hr, μ, and the rate of mRNA synthesis (nucleotides/min/cell), $r_m \times 10^5$

μ	0.6	1.0	1.5	2.0	2.5
r_m	4.3	9.1	13	19	23

Linear Model for Rate of mRNA Synthesis

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- Want to find the best linear model by varying the slope, a

Graph of Data and Best Linear Model

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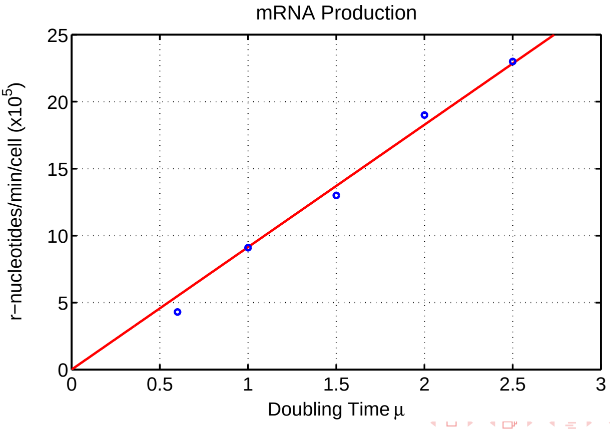

Least Squares Best Fit to Linear Model 1

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Linear model passing through the origin has the form

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- The linear least squares best fit of this model to the data uses only the slope of the model, a
- The sum of the squares of the errors is computed from each of the error terms

$$\begin{aligned} e_1^2 &= (4.3 - 0.6a)^2 \\ e_2^2 &= (9.1 - a)^2 \\ e_3^2 &= (13 - 1.5a)^2 \\ e_4^2 &= (19 - 2a)^2 \\ e_5^2 &= (23 - 2.5a)^2 \end{aligned}$$

Least Squares Best Fit to Linear Model 2

Sum of Square Errors is given by

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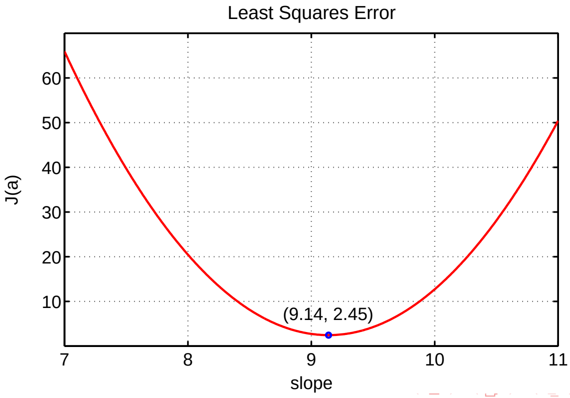
- This occurs the vertex, a_v, of this quadratic equation

Graph of Least Squares Function $J(a)$

Graph of Least Squares Function – **Least Squares Best fit** when a is at a minimum, the vertex $a_v = 9.14$



Definitions and Properties of Functions

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- A function is a relationship between one set of objects and another set of objects with only one possible association in the second set for each member of the first set

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  - Claim that the best model is when this function is at its lowest point

Definition of a Function

Definition: A **function** of a variable x is a rule f that assigns to each value of x a unique number $f(x)$. The variable x is the **independent variable**, and the set of values over which x may vary is called the **domain** of the function. The set of values $f(x)$ over the domain gives the **range** of the function

Definition of a Graph

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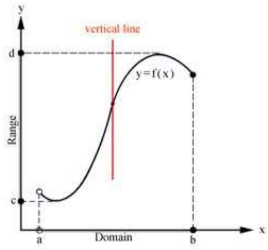
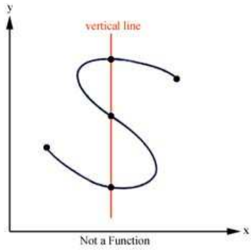
- Often a function is described by a **graph** in the xy-coordinate system
- By convention x is the domain of the function and y is the range of the function
- The **graph** is defined by the set of points $(x, f(x))$ for all x in the domain

Vertical Line Test

The **Vertical Line Test** states that a curve in the xy-plane is the graph of a function if and only if each vertical line touches the curve *at no more than one point*



Domain: $a < x \leq b$
Range: $c \leq y \leq d$



Example of Domain and Range 1

Example 1: Consider the function

$$f(t) = t^2 - 1$$

Skip Example

a. What is the range of $f(t)$ (assuming a domain of all t)?

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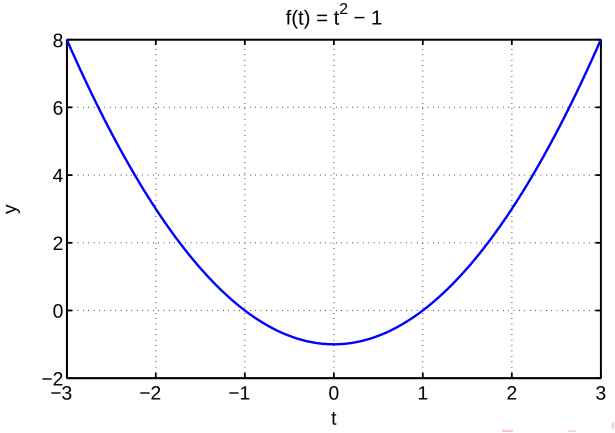
a. What is the range of $f(t)$ (assuming a domain of all t)?

Solution a: $f(t)$ is a parabola with its vertex at $(0, -1)$ pointing up.

Since the vertex is the low point of the function, it follows that range of $f(t)$ is $-1 \leq y < \infty$

Graph of Example 1 2

Graph for the domain and range of $f(t)$



Example of Domain and Range

Example 1 (cont): More on the function

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b. Find the domain of $f(t)$, if the range of f is restricted to $f(t) < 0$

Example of Domain and Range 3

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It follows that the domain is $-1 < t < 1$

Addition and Multiplication of Functions

Example 2: Let $f(x) = x - 1$ and $g(x) = x^2 + 2x - 3$

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The multiplication of the two functions

$$\begin{aligned} f(x)g(x) &= (x-1)(x^2+2x-3) \\ &= x^3 + 2x^2 - 3x - x^2 - 2x + 3 \\ &= x^3 + x^2 - 5x + 3 \end{aligned}$$

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$$f(x) + g(x) = \frac{3}{x-6} + \frac{-2}{x+2} = \frac{3(x+2) - 2(x-6)}{(x-6)(x+2)}$$

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$$\begin{aligned} f(x) + g(x) &= \frac{3}{x-6} + \frac{-2}{x+2} = \frac{3(x+2) - 2(x-6)}{(x-6)(x+2)} \\ &= \frac{x+18}{x^2 - 4x - 12} \end{aligned}$$

Composition of Functions

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Given functions $f(x)$ and $g(x)$, the composite $f(g(x))$ is formed by inserting $g(x)$ wherever x appears in $f(x)$

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Given functions $f(x)$ and $g(x)$, the composite $f(g(x))$ is formed by inserting $g(x)$ wherever x appears in $f(x)$

Note that the domain of the composite function is the range of $g(x)$

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Solution: For the first composite function

$$f(g(x)) = 3(x^2 - 2x + 3) + 2 = 3x^2 - 6x + 11$$

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The second composite function

$$g(f(x)) = (3x + 2)^2 - 2(3x + 2) + 3 = 9x^2 + 6x + 3$$

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$$g(f(x)) = (3x + 2)^2 - 2(3x + 2) + 3 = 9x^2 + 6x + 3$$

Clearly, $f(g(x)) \neq g(f(x))$

Even and Odd Functions

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2. **Odd** if $f(x) = -f(-x)$ for all x in the domain of f. In this case, the graph is symmetrical with respect to the origin

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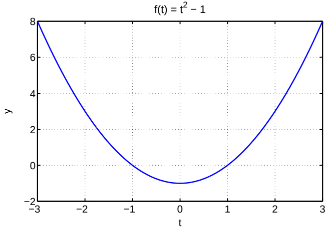
The Graph of an Even Function is symmetric about the y-axis



One-to-One Function

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Equivalently, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Inverse Functions

Definition: If a function f is **one-to-one**, then its corresponding **inverse function**, denoted f^{-1}, satisfies:

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$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Since these are composite functions, the domains of f and f^{-1} are restricted to the ranges of f^{-1} and $f(x)$, respectively

Example of an Inverse Function 1

Consider the function

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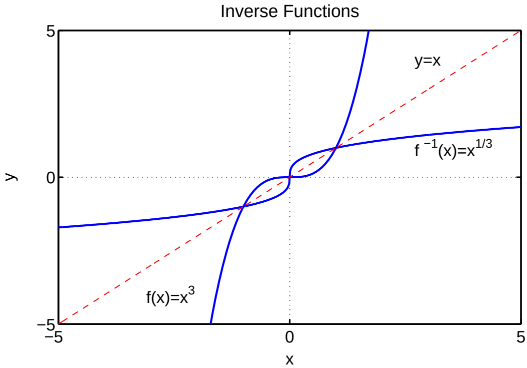
The domain and range for these functions are all of x

$$f^{-1}(f(x)) = \left(x^3\right)^{1/3} = x = \left(x^{1/3}\right)^3 = f(f^{-1}(x))$$

Example of an Inverse Function 2



These functions are mirror images through the line $y = x$ (**the Identity Map**)