Calculus for the Life Sciences I

Lecture Notes – Chain Rule

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Outline

Chain Rule

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- Functional relationships where one measurable quantity depends on another, while the second quantity is a function of a third quantity

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- This functional relationship is a **composite function**
- The differentiation of a composite function requires the **chain rule**

Average Height and Weight of Girls 1

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- Height and age are approximated well by a **linear function**
- Height and weight of animals satisfies an **allometric model**

Average Height and Weight of Girls

Average Height and Weight of American Girls

age (years)	height (cm)	weight (kg)	age (years)	height (cm)	weight (kg)
1	75	9.5	8	126	25.0
2	87	11.8	9	132	29.1
3	94	15.0	10	138	32.7
4	102	15.9	11	144	37.3
5	108	18.2	12	151	41.4
6	114	20.0	13	156	46.8
7	121	21.8			

Average Height and Weight of Girls 3

Least Squares Best Fit: Model of Height as a function of age

$$h(a) = 6.45\,a + 73.9$$

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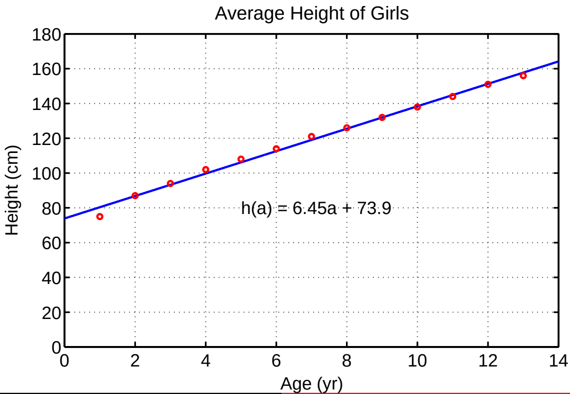
Model shows that the average girl grows about **6.45 cm/yr**



Average Height and Weight of Girls 4

Allometric Model: An allometric model for the height and weight of a girl satisfies

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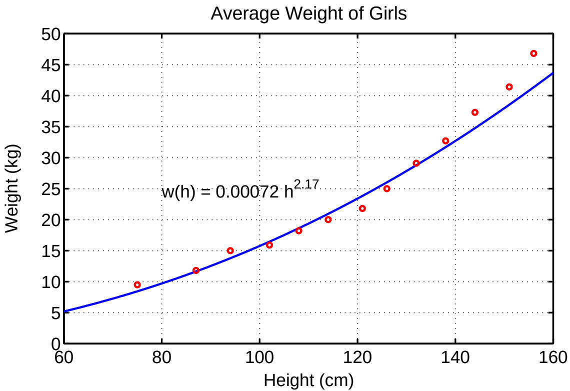

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Composite Model: The **linear model** shows that the average girl grows about **6.45 cm/yr**

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  - Create a **composite function** of the **allometric model** and the **linear model** to give a function of the weight as a function of age

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- How do we find the **rate of change in weight** for a girl at any particular age (between 1 and 13)?
  - The **Allometric Model** gives the weight as a function of height
  - Create a **composite function** of the **allometric model** and the **linear model** to give a function of the weight as a function of age
  - The **chain rule** gives the rate of change of weight with respect to age

Chain Rule

Chain Rule: Consider the **composite function** $f(g(x))$

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- Alternately, the chain rule is written

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x)$$

Example – Chain Rule

Example - Chain Rule: Consider the the function

$$h(x) = (x^2 + 2x - 5)^5$$

Find $h'(x)$

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Solution: Consider the composite of the functions

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From the **chain rule**

$$h'(x) \;=\; 5(g(x))^4(2x+2)$$

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$$\begin{aligned} h'(x) &= 5(g(x))^4(2x + 2) \\ h'(x) &= 5(x^2 + 2x - 5)^4(2x + 2) \end{aligned}$$

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From the **chain rule**

$$h'(x) = 6(x^3 - 4x^2 + e^{-2x})^5(3x^2 - 8x - 2e^{-2x})$$

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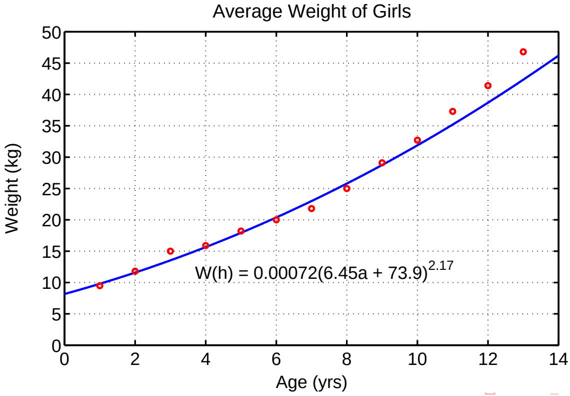
- The composite function weight as a function of age

$$W(a) = 0.000720(6.45\, a + 73.9)^{2.17}$$

Rate of Change in Weight 2

Composite Function: Weight as a function of age

$$W(a) = 0.000720(6.45\,a + 73.9)^{2.17}$$



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Rate of Change in Weight 3

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From the chain rule, the derivative of the weight function is

$$\frac{dW}{da} = \frac{dW}{dh} \cdot \frac{dh}{da}$$

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$$\frac{dW}{dh} = 2.17(0.000720)h^{1.17} \quad \text{and} \quad \frac{dh}{da} = 6.45$$

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$$\frac{dW}{dh} = 2.17(0.000720)h^{1.17} \quad \text{and} \quad \frac{dh}{da} = 6.45$$

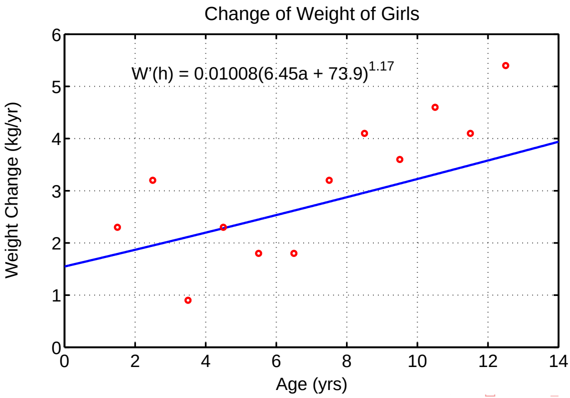
Combining these and substituting the expression for h

$$W'(a) = 0.01008(6.45\,a + 73.9)^{1.17}$$

Rate of Change in Weight

Change of Weight of Girls Graph of

$$W'(a) = 0.01008(6.45\,a + 73.9)^{1.17}$$



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Rate of Change in Weight

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- This graph is almost linear, since it is to the 1.17 power
- The actual average weight changes are given for the data above
- We see that the model underpredicts the weight gain for older girls

Normal Distribution 1

Normal Distribution: This is an important function in **statistics**

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  - σ is the **standard deviation**
  - μ is the **mean** of the distribution

Normal Distribution 2

Normal Distribution: Consider

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- Plot this function for several values of σ

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- Plot this function for several values of σ
- Discuss the importance of the results

Normal Distribution

Solution: Consider

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Normal Distribution 3

Solution: Consider

$$N(x) = \frac{a}{\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

The derivative is

$$\frac{dN}{dx} = \frac{a}{\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \left(-\frac{2(x-\mu)}{2\sigma^2} \right)$$

Normal Distribution 3

Solution: Consider

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The derivative is zero when $x = \mu$, so there is a maximum at $(\mu, \frac{a}{\sigma})$

Normal Distribution

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The second derivative is

$$\frac{d^2N}{dx^2} = \frac{a}{\sigma^3}\left((x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}\left(-\frac{2(x-\mu)}{2\sigma^2}\right) + e^{-(x-\mu)^2/(2\sigma^2)} \cdot 1\right)$$

Normal Distribution 4

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$$\frac{d^2N}{dx^2} = \frac{a}{\sigma^3}\left(1 - \frac{(x-\mu)^2}{\sigma^2}\right) e^{-(x-\mu)^2/(2\sigma^2)}$$

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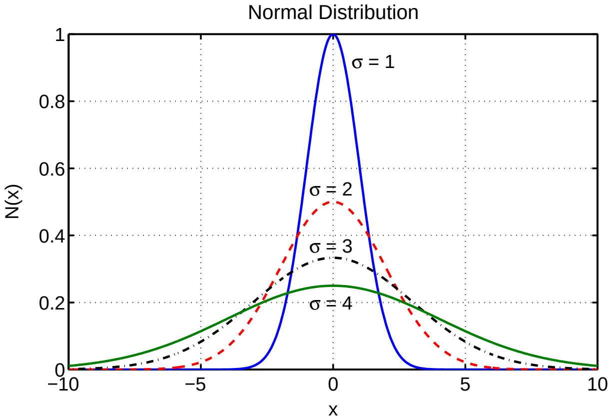
The points of inflection occur at $x = \mu \pm \sigma$ with

$$N(\mu \pm \sigma) = \frac{a}{\sigma} e^{-\frac{1}{2}}$$

Normal Distribution

Solution: Graph of the **Normal Distribution** with $\mu = 0$ and $\sigma = 1, 2, 3, 4$



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- It can be shown that 68% of the area under the normal distribution occurs in the interval, $[-\sigma, \sigma]$
- The **area** under a **distribution function** is important in measuring **probabilities** and **confidence intervals** for statistics

Hassell's Model 1

Hassell's Model is used in the study of insect populations

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Suppose that a study shows that a population, P_n, of butterflies satisfies the nonlinear discrete dynamic model given by:

$$P_{n+1} = H(P_n) = \frac{81\,P_n}{(1 + 0.002\,P_n)^4}$$

where n is in weeks

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- Let $P_0 = 200$, then find P_1 and P_2
- Find the intercepts, all extrema of $H(P)$, and any asymptotes for $P \geq 0$
- Graph $H(P)$

Hassell's Model 1

Hassell's Model is used in the study of insect populations

Suppose that a study shows that a population, P_n, of butterflies satisfies the nonlinear discrete dynamic model given by:

$$P_{n+1} = H(P_n) = \frac{81\, P_n}{(1 + 0.002\, P_n)^4}$$

where n is in weeks

- Let $P_0 = 200$, then find P_1 and P_2
- Find the intercepts, all extrema of $H(P)$, and any asymptotes for $P \geq 0$
- Graph $H(P)$
- Determine the equilibria and analyze the behavior of the solution near the equilibria

Hassell's Model

Solution: For Hassell's model

$$P_{n+1} = H(P_n) = \frac{81\, P_n}{(1 + 0.002\, P_n)^4}$$

Hassell's Model 2

Solution: For Hassell's model

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- Let $P_0 = 200$

Hassell's Model 2

Solution: For Hassell's model

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- Let $P_0 = 200$
  - $P_1 = H(200) = \dfrac{16200}{(1.4)^4} = 4271$

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- Let $P_0 = 200$
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  - $P_2 = H(4271) = 43$
- This dramatic population swing suggests an instability

Hassell's Model 3

Solution (cont): The **Updating function** is

$$H(P) = \frac{81\,P}{(1 + 0.002\,P)^4}$$

Hassell's Model

3

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- For $H(P)$, the only intercept is $(0, 0)$

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- Differentiate the function

$$\frac{dH}{dP} = 81\frac{(1+0.002\,P)^4\cdot 1 - P\cdot 4(1+0.002\,P)^3 0.002}{(1+0.002\,P)^8}$$

$$\frac{dH}{dP} = 81\frac{(1-0.006\,P)}{(1+0.002\,P)^5}$$

Hassell's Model

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- **Critical points** satisfy $H'(P) = 0$, so

$$1 - 0.006\,P = 0 \quad \text{or} \quad P = \frac{500}{3} = 166.7$$

Hassell's Model 4

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- **Critical points** satisfy $H'(P) = 0$, so

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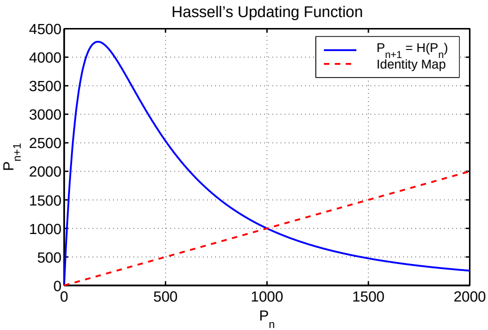
- With $H(500/3) = 4271.5$, the maximum occurs at

$$(166.7, 4271.5)$$

Hassell's Model

Solution (cont): The graph of

$$H(P) = \frac{81\,P}{(1 + 0.002\,P)^4}$$



Hassell's Model

Equilibria for Hassell's Model: The equilibria satisfy $P_{n+1} = P_n = P_e$, so

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Hassell's Model 6

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It follows that one equilibrium is $P_e = 0$ and

$$\begin{aligned} (1+0.002\,P_e)^4 &= 81 \\ 1+0.002\,P_e &= 3 \\ P_e &= 1000 \end{aligned}$$

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- At $P_e = 1000$, $H'(1000) = 81(-5)/243 = -5/3 < -1$

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- This model produces a **period 4** solution with the solution asymptotically oscillating from $163 \to 4271 \to 42 \to 2453$

Hassell's Model

Simulation: Starting $P_0 = 200$, we simulate

$$P_{n+1} = \frac{81\,P_n}{(1 + 0.002\,P_n)^4}$$