Calculus for the Life Sciences I

Lecture Notes – Chain Rule

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Outline

Chain Rule

Chain Rule

- Functional relationships where one measurable quantity depends on another, while the second quantity is a function of a third quantity
- This functional relationship is a **composite function**
- The differentiation of a composite function requires the **chain rule**

Average Height and Weight of Girls 1

Average Height and Weight of Girls

- Over a range of ages the rate of growth of girls in height is constant
- Height and age are approximated well by a **linear function**
- Height and weight of animals satisfies an **allometric model**

Average Height and Weight of Girls

Average Height and Weight of American Girls

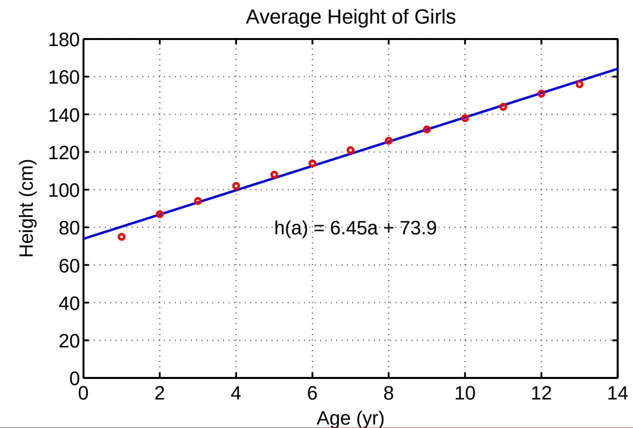
age (years)	height (cm)	weight (kg)	age (years)	height (cm)	weight (kg)
1	75	9.5	8	126	25.0
2	87	11.8	9	132	29.1
3	94	15.0	10	138	32.7
4	102	15.9	11	144	37.3
5	108	18.2	12	151	41.4
6	114	20.0	13	156	46.8
7	121	21.8			

Average Height and Weight of Girls

Least Squares Best Fit: Model of Height as a function of age

$$h(a) = 6.45\,a + 73.9$$

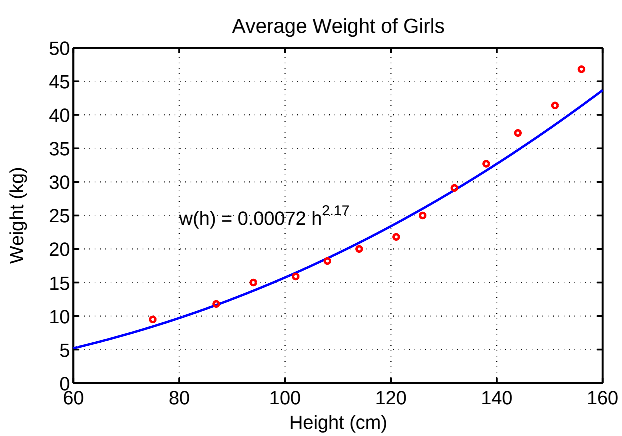
Model shows that the average girl grows about **6.45 cm/yr**



Average Height and Weight of Girls

Allometric Model: An allometric model for the height and weight of a girl satisfies

$$W(h) = 0.000720\,h^{2.17}$$



Average Height and Weight of Girls

Composite Model: The **linear model** shows that the average girl grows about **6.45 cm/yr**

- How do we find the **rate of change in weight** for a girl at any particular age (between 1 and 13)?
  - The **Allometric Model** gives the weight as a function of height
  - Create a **composite function** of the **allometric model** and the **linear model** to give a function of the weight as a function of age
  - The **chain rule** gives the rate of change of weight with respect to age

Chain Rule

Chain Rule: Consider the **composite function** $f(g(x))$

- Suppose that both $f(u)$ and $u = g(x)$ are differentiable functions
- The **chain rule** for differentiation of this composite function is given by

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

- Alternately, the chain rule is written

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

Example – Chain Rule

Example - Chain Rule: Consider the the function

$$h(x) = (x^2 + 2x - 5)^5$$

Find $h'(x)$

Solution: Consider the composite of the functions

$$f(u) = u^5 \quad \text{and} \quad g(x) = x^2 + 2x - 5$$

The derivatives of both f and g are

$$f'(u) = 5\,u^4 \quad \text{and} \quad g'(x) = 2x + 2$$

From the **chain rule**

$$\begin{aligned} h'(x) &= 5(g(x))^4(2x+2) \\ h'(x) &= 5(x^2+2x-5)^4(2x+2) \end{aligned}$$

Example 2 – Chain Rule

Example 2 - Chain Rule: Consider the the function

$$h(x) = e^{2-x^2}$$

Find $h'(x)$

Solution: Consider the composite of the functions

$$f(u) = e^u \quad \text{and} \quad g(x) = 2 - x^2$$

The derivatives of both f and g are

$$f'(u) = e^u \quad \text{and} \quad g'(x) = -2x$$

From the **chain rule**

$$h'(x) = e^{2-x^2}(-2x)$$

Example 3 – Chain Rule

Example 3 - Chain Rule: Consider the the function

$$h(x) = (x^3 - 4x^2 + e^{-2x})^6$$

Find $h'(x)$

Skip Example

Solution: Consider the composite of the functions

$$f(u) = u^6 \quad \text{and} \quad g(x) = x^3 - 4x^2 + e^{-2x}$$

The derivatives of both f and g are

$$f'(u) = 6u^5 \quad \text{and} \quad g'(x) = 3x^2 - 8x - 2e^{-2x}$$

From the **chain rule**

$$h'(x) = 6(x^3 - 4x^2 + e^{-2x})^5(3x^2 - 8x - 2e^{-2x})$$

Rate of Change in Weight 1

Rate of Change in Weight: The example for the weight and height of a child given above found

- The weight W as a function of height h is

$$W(h) = 0.000720\, h^{2.17}$$

- The height as a function of age is

$$h(a) = 6.45\, a + 73.9$$

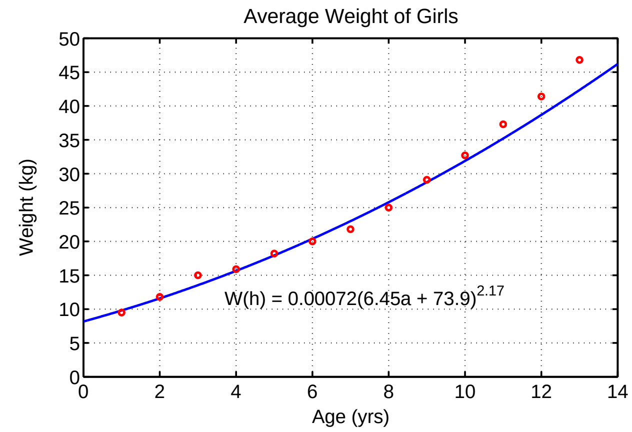
- The composite function weight as a function of age

$$W(a) = 0.000720(6.45\, a + 73.9)^{2.17}$$

Rate of Change in Weight 2

Composite Function: Weight as a function of age

$$W(a) = 0.000720(6.45\, a + 73.9)^{2.17}$$



Rate of Change in Weight 3

Chain Rule: Weight as a function of age

$$W(a) = 0.000720(6.45\, a + 73.9)^{2.17}$$

From the chain rule, the derivative of the weight function is

$$\frac{dW}{da} = \frac{dW}{dh} \cdot \frac{dh}{da}$$

$$\frac{dW}{dh} = 2.17(0.000720)h^{1.17} \quad \text{and} \quad \frac{dh}{da} = 6.45$$

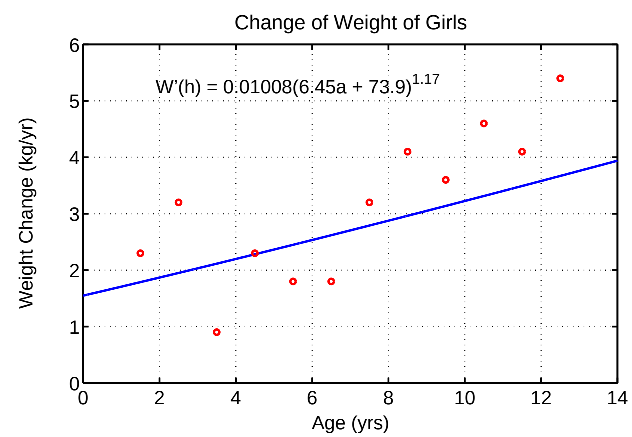
Combining these and substituting the expression for h

$$W'(a) = 0.01008(6.45\, a + 73.9)^{1.17}$$

Rate of Change in Weight 4

Change of Weight of Girls Graph of

$$W'(a) = 0.01008(6.45\, a + 73.9)^{1.17}$$



Rate of Change in Weight 5

Change of Weight of Girls Graph of

$$W'(a) = 0.01008(6.45\,a + 73.9)^{1.17}$$

- This graph is almost linear, since it is to the 1.17 power
- The actual average weight changes are given for the data above
- We see that the model underpredicts the weight gain for older girls

Normal Distribution 1

Normal Distribution: This is an important function in **statistics**

- Gives the classic **Bell curve**
- The **normal distribution function** is

$$N(x) = \frac{a}{\sigma}e^{-(x-\mu)^2/(2\sigma^2)}$$

  - a is the normalizing factor
  - σ is the **standard deviation**
  - μ is the **mean** of the distribution

Normal Distribution 2

Normal Distribution: Consider

$$N(x) = \frac{a}{\sigma}e^{-(x-\mu)^2/(2\sigma^2)}$$

- Find the maximum and points of inflection
- Plot this function for several values of σ
- Discuss the importance of the results

Normal Distribution 3

Solution: Consider

$$N(x) = \frac{a}{\sigma}e^{-(x-\mu)^2/(2\sigma^2)}$$

The derivative is

$$\frac{dN}{dx} = \frac{a}{\sigma}e^{-(x-\mu)^2/(2\sigma^2)}\left(-\frac{2(x-\mu)}{2\sigma^2}\right)$$
$$\frac{dN}{dx} = -\frac{a(x-\mu)}{\sigma^3}e^{-(x-\mu)^2/(2\sigma^2)}$$

The derivative is zero when $x = \mu$, so there is a maximum at $(\mu, \frac{a}{\sigma})$

Normal Distribution 4

Solution: The derivative is

$$\frac{dN}{dx} = -\frac{a(x-\mu)}{\sigma^3} e^{-(x-\mu)^2/(2\sigma^2)}$$

The second derivative is

$$\frac{d^2N}{dx^2} = \frac{a}{\sigma^3}\left((x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}\left(-\frac{2(x-\mu)}{2\sigma^2}\right) + e^{-(x-\mu)^2/(2\sigma^2)} \cdot 1\right)$$

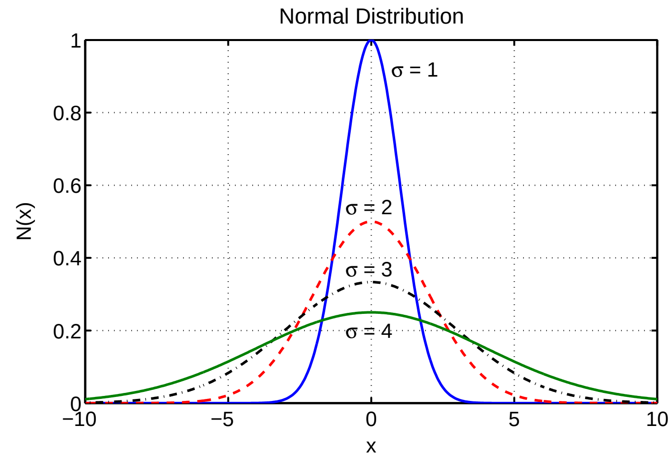
$$\frac{d^2N}{dx^2} = \frac{a}{\sigma^3}\left(1 - \frac{(x-\mu)^2}{\sigma^2}\right) e^{-(x-\mu)^2/(2\sigma^2)}$$

The points of inflection occur at $x = \mu \pm \sigma$ with

$$N(\mu \pm \sigma) = \frac{a}{\sigma} e^{-\frac{1}{2}}$$

Normal Distribution 5

Solution: Graph of the **Normal Distribution** with $\mu = 0$ and $\sigma = 1, 2, 3, 4$



Normal Distribution 6

Properties of Normal Distribution

$$N(x) = \frac{a}{\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

- As noted above, the **mean** of the normal distribution is μ
- The normal distribution is a **bell-shaped** curve centered about its mean
- The points of inflection occur one standard deviation, σ, from the mean, μ
- It can be shown that 68% of the area under the normal distribution occurs in the interval, $[-\sigma, \sigma]$
- The **area** under a **distribution function** is important in measuring **probabilities** and **confidence intervals** for statistics

Hassell's Model 1

Hassell's Model is used in the study of insect populations

Suppose that a study shows that a population, P_n, of butterflies satisfies the nonlinear discrete dynamic model given by:

$$P_{n+1} = H(P_n) = \frac{81\,P_n}{(1 + 0.002\,P_n)^4}$$

where n is in weeks

- Let $P_0 = 200$, then find P_1 and P_2
- Find the intercepts, all extrema of $H(P)$, and any asymptotes for $P \geq 0$
- Graph $H(P)$
- Determine the equilibria and analyze the behavior of the solution near the equilibria

Hassell's Model 2

Solution: For Hassell's model

$$P_{n+1} = H(P_n) = \frac{81\,P_n}{(1+0.002\,P_n)^4}$$

- Let $P_0 = 200$
  - $P_1 = H(200) = \dfrac{16200}{(1.4)^4} = 4271$
  - $P_2 = H(4271) = 43$
- This dramatic population swing suggests an instability

Hassell's Model 3

Solution (cont): The **Updating function** is

$$H(P) = \frac{81\,P}{(1+0.002\,P)^4}$$

- For $H(P)$, the only intercept is $(0,0)$
- The power of P in the denominator (4) exceeds the power in the numerator (1), so there is a horizontal asymptote with $H = 0$
- Differentiate the function

$$\frac{dH}{dP} = 81\frac{(1+0.002\,P)^4\cdot 1 - P\cdot 4(1+0.002\,P)^3 0.002}{(1+0.002\,P)^8}$$

$$\frac{dH}{dP} = 81\frac{(1-0.006\,P)}{(1+0.002\,P)^5}$$

Hassell's Model 4

Solution (cont): The derivative is

$$\frac{dH}{dP} = 81\frac{(1-0.006\,P)}{(1+0.002\,P)^5}$$

- **Critical points** satisfy $H'(P) = 0$, so

$$1 - 0.006\,P = 0 \quad \text{or} \quad P = \frac{500}{3} = 166.7$$

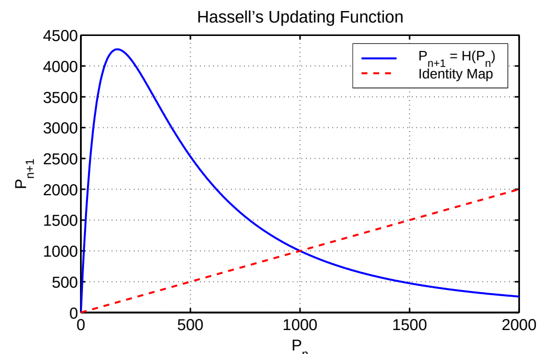
- With $H(500/3) = 4271.5$, the maximum occurs at

$$(166.7, 4271.5)$$

Hassell's Model 5

Solution (cont): The graph of

$$H(P) = \frac{81\,P}{(1+0.002\,P)^4}$$



Hassell's Model 6

Equilibria for Hassell's Model: The equilibria satisfy $P_{n+1} = P_n = P_e$, so

$$\begin{aligned} P_e &= \frac{81\,P_e}{(1+0.002\,P_e)^4} \\ P_e(1+0.002\,P_e)^4 &= 81\,P_e \end{aligned}$$

It follows that one equilibrium is $P_e = 0$ and

$$\begin{aligned} (1+0.002\,P_e)^4 &= 81 \\ 1+0.002\,P_e &= 3 \\ P_e &= 1000 \end{aligned}$$

Hassell's Model 7

Stability of Equilibria: The equilibria are $P_e = 0$ and 1000

The derivative is given by

$$\frac{dH}{dP} = 81\frac{(1-0.006\,P)}{(1+0.002\,P)^5}$$

- At $P_e = 0$, $H'(0) = 81 > 1$, which implies that solutions monotonically grow away from 0
- At $P_e = 1000$, $H'(1000) = 81(-5)/243 = -5/3 < -1$
- We will show that this implies that the solution near this equilibrium oscillates and goes away from the equilibrium
- This model produces a **period 4** solution with the solution asymptotically oscillating from $163 \to 4271 \to 42 \to 2453$

Hassell's Model 8

Simulation: Starting $P_0 = 200$, we simulate

$$P_{n+1} = \frac{81\,P_n}{(1+0.002\,P_n)^4}$$