Calculus for the Life Sciences I

Lecture Notes – Applications of the Derivative

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Spring 2013

Outline

Applications of the Derivative

Applications of the Derivative

- Sketching Graphs

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- Sketching Graphs
- Finding Maxima and Minima

Applications of the Derivative

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- Sketching Graphs
- Finding Maxima and Minima
- Optimal Values

Applications of the Derivative

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- Sketching Graphs
- Finding Maxima and Minima
- Optimal Values
- Points of Inflection or Steepest parts of a Function

Menstrual Cycle

Body Temperature Fluctuation during the Menstrual Cycle

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- Mammals regulate their body temperature in a narrow range to maintain optimal physiological responses

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- Variations include
  - Circadian rhythms (a few tenths of a degree Celsius)

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Body Temperature Fluctuation during the Menstrual Cycle

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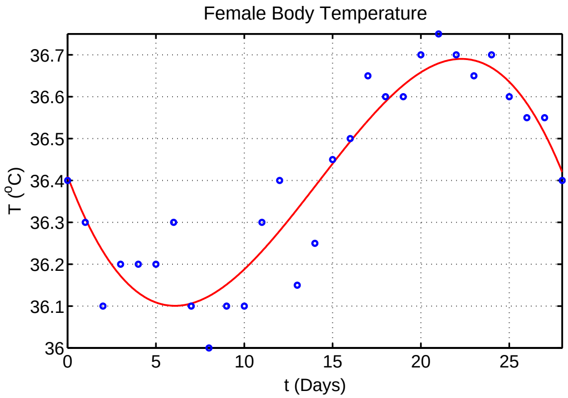
- Variations in body temperature occur during exercise, stress, infection, and other normal situations
- Neurological control of the temperature
- Variations include
  - Circadian rhythms (a few tenths of a degree Celsius)
  - Menstrual cycle - Ovulation often corresponds to the sharpest rise in temperature

Female Basal Body Temperature 1

Female Basal Body Temperature



Female Basal Body Temperature 2

Female Basal Body Temperature

- The best cubic polynomial fitting the data above is

$$T(t) = -0.0002762\, t^3 + 0.01175\, t^2 - 0.1121\, t + 36.41$$

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- Want to find the high and low temperatures
- Determine the time of peak fertility when the temperature is rising most rapidly

Female Basal Body Temperature

Maximum and Minimum for

$$T(t) = -0.0002762\, t^3 + 0.01175\, t^2 - 0.1121\, t + 36.41$$

- The high and low temperatures occur when the curve has slope of zero

Female Basal Body Temperature 3

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- The derivative of the temperature is

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- Note that the derivative is a different function from the original function

Female Basal Body Temperature

Maximum and Minimum

- The roots of the derivative (quadratic equation) are

$$t = 6.069 \quad \text{and} \quad 22.29 \text{ days}$$

Female Basal Body Temperature 4

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- There is only a $0.6^\circ C$ difference between the high and low basal body temperature during a 28 day menstrual cycle by the approximating function

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- There is only a $0.6^\circ C$ difference between the high and low basal body temperature during a 28 day menstrual cycle by the approximating function
- The data varied by $0.75^\circ C$

Female Basal Body Temperature 5

Maximum Increase in Temperature

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Female Basal Body Temperature

Maximum Increase in Temperature

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- The maximum occurs at the midpoint between the roots of the quadratic equation
- Alternately, the derivative of $T'(t)$ or the second derivative of $T(t)$ equal to zero gives the maximum
- The second derivative is

$$T''(t) = -0.0016572\,t + 0.02350$$

Female Basal Body Temperature

Maximum Increase in Temperature

- The second derivative is zero at the **Point of Inflection** at $t = 14.18$ with $T(14.18) = 36.4°C$

Female Basal Body Temperature 6

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- The maximum rate of change in body temperature is

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- This model suggests that the peak fertility occurs on **day 14**, which is consistent with what is known about ovulation

Maxima, Minima, and Critical Points 1

Critical Points, Increasing and Decreasing

The derivative is zero at critical points for the graph of a **smooth function**

Maxima, Minima, and Critical Points 1

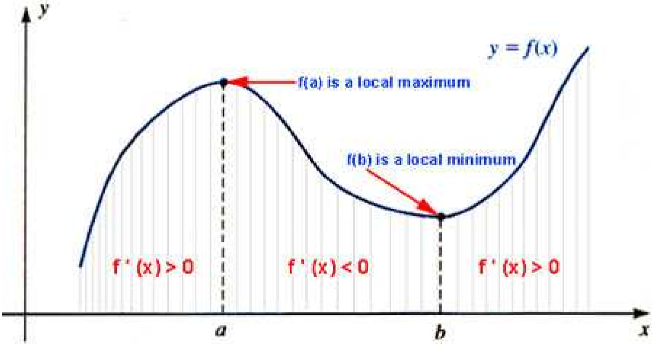
Critical Points, Increasing and Decreasing

The derivative is zero at critical points for the graph of a **smooth function**

Definition: A smooth function $f(x)$ is said to be increasing on an interval (a,b) if $f'(x) > 0$ for all $x \in (a,b)$. Similarly, a smooth function $f(x)$ is said to be decreasing on an interval (a,b) if $f'(x) < 0$ for all $x \in (a,b)$

Maxima, Minima, and Critical Points 2

Maxima, Minima, and Critical Points



Maxima, Minima, and Critical Points

Maxima and Minima

- A high point of the graph is where $f(x)$ changes from increasing to decreasing

Maxima, Minima, and Critical Points 3

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- A low point on a graph is where $f(x)$ changes from decreasing to increasing

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- If $f(x)$ is a smooth function, then either case has the derivative passing through zero

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Definition: A smooth function $f(x)$ is said to have a **local** or **relative maximum** at a point c, if $f'(c) = 0$ and $f'(x)$ changes from positive to negative for values of x near c. Similarly, a smooth function $f(x)$ is said to have a **local** or **relative minimum** at a point c, if $f'(c) = 0$ and $f'(x)$ changes from negative to positive for values of x near c.

Maxima, Minima, and Critical Points

Critical Points

Definition: If $f(x)$ is a smooth function with $f'(x_c) = 0$, then x_c is said to be a **critical point** of $f(x)$

Maxima, Minima, and Critical Points 4

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Maxima, Minima, and Critical Points

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- Critical points help find the local high and low points on a graph
- Some critical points are neither maxima or minima

Example – Graphing a Polynomial 1

Graphing a Polynomial Consider

$$f(x) = x^3 - 6\,x^2 - 15\,x + 2$$

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- Find critical points, maxima, and minima

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Graphing a Polynomial Consider

$$f(x) = x^3 - 6x^2 - 15x + 2$$

- Find critical points, maxima, and minima
- Sketch a graph of the function

Example – Graphing a Polynomial 2

With the function

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we have the derivative

Example – Graphing a Polynomial 2

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$$f'(x) = 3\,x^2 - 12\,x - 15 = 3(x+1)(x-5)$$

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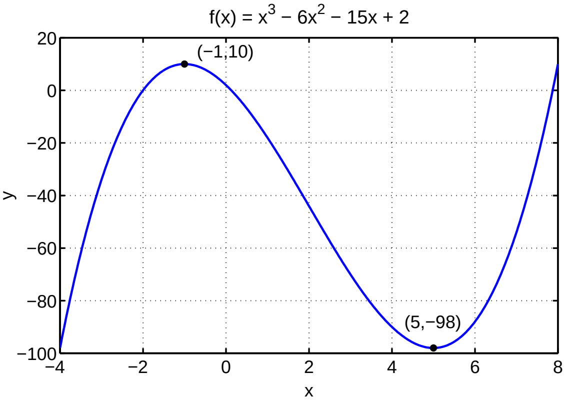
- Since $f(5) = -98$, a **local minimum** occurs at $(5, -98)$
- The y-intercept is $(0, 2)$

Example – Graphing a Polynomial 3

This gives enough for a reasonable sketch of the graph
Note the x-intercepts are very hard to find



Second Derivative and Concavity

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- The second derivative is a measure of the concavity of a function
- For smooth functions, the **maxima** generally occur where the function is **concave downward**, while **minima** occur where the function is **concave upward**

Second Derivative Test

Second Derivative Test: Let $f(x)$ be a smooth function. Suppose that $f'(x_c) = 0$, so x_c is a critical point of f. If $f''(x_c) < 0$, then x_c is a **relative maximum**. If $f''(x_c) > 0$, then x_c is a **relative minimum**.

If $f''(x_c) = 0$, then we get **no information** about the function at the critical point x_c

Example – Graphing a Polynomial 4

Continuing the example $f(x) = x^3 - 6\,x^2 - 15\,x + 2$

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- The second derivative at the critical point $x_c = -1$ gives

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- The second derivative at the critical point $x_c = 5$ gives

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- The function is concave upward, so this is a relative minimum

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- A point of inflection is where the derivative function has a maximum or minimum, so the function is increasing or decreasing most rapidly
- From an applications point of view, if the function is describing a population, then the point of inflection would be where the population is increasing or decreasing most rapidly
- The point of inflection measures when the change of a function is its greatest or smallest

Example – Graphing a Polynomial 5

Continuing the example

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- The point of inflection is $f''(x) = 0$, which is when $x = 2$

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- The second derivative is

$$f''(x) = 6\,x - 12$$

- The point of inflection is $f''(x) = 0$, which is when $x = 2$
- The point of inflection occurs at $(2, -44)$

Example – Graphing a Cubic Polynomial 1

Graphing a Polynomial: Consider

$$y(x) = 12\,x - x^3$$

Skip Example

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Graphing a Polynomial: Consider

$$y(x) = 12\,x - x^3$$

- Find the intercepts
- Find critical points, maxima, and minima
- Find the points of inflection
- Sketch the graphs of the function, its derivative, and the second derivative

Skip Example

Example – Graphing a Cubic Polynomial 2

The ***y*-intercept** for $y(x) = 12\,x - x^3$ is **(0,0)**

Example – Graphing a Cubic Polynomial 2

The **y-intercept** for $y(x) = 12\,x - x^3$ is **(0,0)**

The **x-intercepts** satisfy

$$x(12 - x^2) = 0$$

Example – Graphing a Cubic Polynomial 2

The **y-intercept** for $y(x) = 12\,x - x^3$ is **(0,0)**

The **x-intercepts** satisfy

$$x(12 - x^2) = 0$$

$$x = 0, \pm 2\sqrt{3}$$

Example – Graphing a Cubic Polynomial 3

Maxima and Minima: The **derivative** of $y(x) = 12\,x - x^3$ is

$$y'(x) = 12 - 3\,x^2 = -3(x^2 - 4)$$

Example – Graphing a Cubic Polynomial 3

Maxima and Minima: The **derivative** of $y(x) = 12\,x - x^3$ is

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The **critical points** satisfy $y'(x_c) = 0$, so

$$x_c = -2 \quad \text{and} \quad x_c = 2$$

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- Evaluating the original function at the critical points

$$y(-2) = -16 \quad \text{and} \quad y(2) = 16$$

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$$x_c = -2 \quad \text{and} \quad x_c = 2$$

- Evaluating the original function at the critical points

$$y(-2) = -16 \quad \text{and} \quad y(2) = 16$$

- The extrema for the function are $(-2, -16)$ and $(2, 16)$
- Clearly, $(-2, -16)$ is a **minimum** and $(2, 16)$ is a **maximum**

Example – Graphing a Cubic Polynomial 4

Point of Inflection: The **second derivative** of $y(x) = 12\,x - x^3$ is

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- The **second derivative test** gives
  - $y''(-2) = 12$ is concave upward, so $x_c = -2$ is a minimum
  - $y''(2) = -12$ is concave downward, so $x_c = 2$ is a maximum

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  - $y''(2) = -12$ is concave downward, so $x_c = 2$ is a maximum
- The **Point of Inflection** occurs at $x_p = 0$

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Point of Inflection: The **second derivative** of $y(x) = 12x - x^3$ is

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- The **second derivative test** gives
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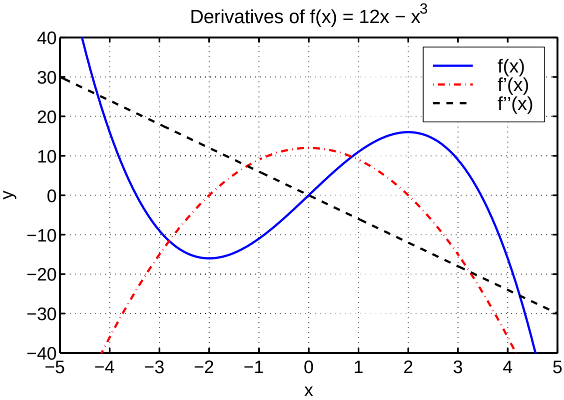
  - $y''(2) = -12$ is concave downward, so $x_c = 2$ is a maximum
- The **Point of Inflection** occurs at $x_p = 0$
- Concavity of the curve changes at $(0, 0)$

Example – Graphing a Cubic Polynomial 5

Graphs of $y(x)$, $y'(x)$, and $y''(x)$



Example – Graphing a Quartic Polynomial 1

Graphing a Polynomial: Consider

$$y(x) = x^4 - 8\,x^2$$

Skip Example

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Graphing a Polynomial: Consider

$$y(x) = x^4 - 8\,x^2$$

- Find the intercepts
- Find critical points, maxima, and minima
- Find the points of inflection
- Sketch the graphs of the function, its derivative, and the second derivative

Skip Example

Example – Graphing a Quartic Polynomial 2

The ***y*-intercept** for $y(x) = x^4 - 8\,x^2$ is **(0,0)**

Example – Graphing a Quartic Polynomial 2

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The **x-intercepts** satisfy

$$x^2(x^2 - 8) = 0$$

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Example – Graphing a Quartic Polynomial 3

Maxima and Minima: The **derivative** of $y(x) = x^4 - 8\,x^2$ is

$$y'(x) = 4\,x^3 - 16\,x = 4\,x(x-2)(x+2)$$

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- Clearly, $(-2,-16)$ and $(2,-16)$ are **minima** and $(0,0)$ is a **maximum**

Example – Graphing a Quartic Polynomial 4

The **second derivative** of $y(x) = x^4 - 8\,x^2$ is

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- The **second derivative test** gives
  - $y''(-2) = 32$ is concave upward, so $x_c = -2$ is a minimum
  - $y''(0) = -16$ is concave downward, so $x_c = 0$ is a maximum
  - $y''(2) = 32$ is concave upward, so $x_c = 2$ is a minimum

Example – Graphing a Quartic Polynomial 5

Points of Inflection: Since the **second derivative** of $y(x) = x^4 - 8\,x^2$ is

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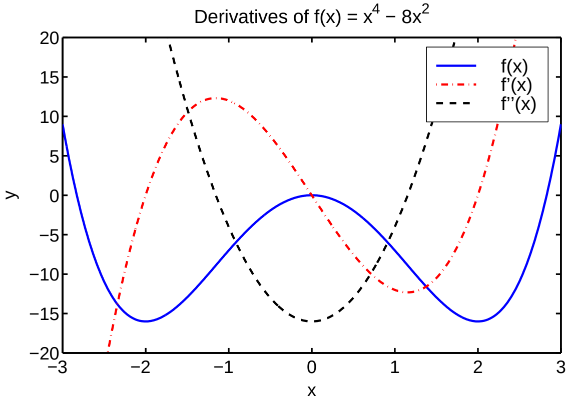
- Inflection points are $x_p \approx (\pm 1.155, -8.889)$

Example – Graphing a Quartic Polynomial 6

Graphs of $y(x)$, $y'(x)$, and $y''(x)$



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Definition: An **absolute minimum** for a function $f(x)$ occurs at a point $x = c$, if $f(c) \leq f(x)$ for all x in the domain of f.

Definition: Similarly, an **absolute maximum** for a function $f(x)$ occurs at a point $x = c$, if $f(c) \geq f(x)$ for all x in the the domain of f.

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Theorem: Suppose that $f(x)$ is a continuous, differentiable function on a closed interval $I = [a, b]$, then $f(x)$ achieves its **absolute minimum** (or **maximum**) on I and its minimum (or maximum) occurs either at a point where $f'(x) = 0$ or at one of the endpoints of the interval

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This **theorem** says to find the function values at all the critical points and the endpoints of the interval, then this small set of values contains the **absolute minimum** and **absolute maximum**

Example – Study of a Population 1

Study of a Population

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Study of a Population

- The ocean water is monitored for fecal contamination by counting certain types of bacteria in a sample of seawater
- Over a week where rain occurred early in the week, data were collected on one type of fecal bacteria
- The population of the particular bacteria (in thousand/cc), $P(t)$, were best fit by the cubic polynomial

$$P(t) = -t^3 + 9\,t^2 - 15\,t + 40,$$

where t is in days

Example – Study of a Population 2

Study of a Population

- Find the rate of change in population per day, dP/dt

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- Find the rate of change in population per day, dP/dt
- What is the rate of change in the population on the third day?
- Find relative and absolute minimum and maximum populations of the bacteria over the time of the surveys
- Determine when the bacterial count is most rapidly increasing
- Sketch a graph of this polynomial fit to the population of bacteria
- When did the rain most likely occur?

Example – Study of a Population 3

Rate of Change in Population: The **derivative** of $P(t) = -t^3 + 9\,t^2 - 15\,t + 40$ is

$$\frac{dP}{dt} = -3\,t^2 + 18\,t - 15$$

Example – Study of a Population 3

Rate of Change in Population: The **derivative** of $P(t) = -t^3 + 9\,t^2 - 15\,t + 40$ is

$$\frac{dP}{dt} = -3\,t^2 + 18\,t - 15$$

Evaluating this on the third day

$$\frac{dP(3)}{dt} = 12(\times 1000/\text{cc}/\text{day})$$

Example – Study of a Population 4

Critical Points: We found

$$\frac{dP}{dt} = -3\,t^2 + 18\,t - 15$$

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  - **Relative Maximum** at $t_c = 5$ with $P(5) = 65(\times 1000/\text{cc})$
- The endpoint values are $P(0) = 40$ and $P(7) = 33$

Example – Study of a Population (4)

Critical Points: We found

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- The **critical points** are
  - $t_c = 1$ and $t_c = 5$
  - **Relative Minimum** at $t_c = 1$ with $P(1) = 33(\times 1000/\text{cc})$
  - **Relative Maximum** at $t_c = 5$ with $P(5) = 65(\times 1000/\text{cc})$
- The endpoint values are $P(0) = 40$ and $P(7) = 33$
- By the theorem above
  - The **absolute maximum** occurs at $t = 5$ with $P(5) = 65$

Example – Study of a Population 4

Critical Points: We found

$$\frac{dP}{dt} = -3\,t^2 + 18\,t - 15 = -3(t-1)(t-5)$$

- The **critical points** are
  - $t_c = 1$ and $t_c = 5$
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  - **Relative Maximum** at $t_c = 5$ with $P(5) = 65(\times 1000/\text{cc})$
- The endpoint values are $P(0) = 40$ and $P(7) = 33$
- By the theorem above
  - The **absolute maximum** occurs at $t = 5$ with $P(5) = 65$
  - The **absolute minimum** occurs at $t = 1$ and 7 with $P(1) = P(7) = 33$

Example – Study of a Population 5

Point of Inflection: The bacteria is increasing most rapidly when the second derivative is zero

Example – Study of a Population

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Since $P'(t) = -3t^2 + 18t - 15$, the second derivative is

$$P''(t) = -6t + 18 = -6(t-3)$$

Example – Study of a Population 5

Point of Inflection: The bacteria is increasing most rapidly when the second derivative is zero

Since $P'(t) = -3t^2 + 18t - 15$, the second derivative is

$$P''(t) = -6t + 18 = -6(t - 3)$$

- The population is increasing most rapidly at $t = 3$ with $P(3) = 49(\times 1000/\text{cc})$

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Since $P'(t) = -3t^2 + 18t - 15$, the second derivative is

$$P''(t) = -6t + 18 = -6(t - 3)$$

- The population is increasing most rapidly at $t = 3$ with $P(3) = 49(\times 1000/\text{cc})$
- This maximum increase is

$$P'(3) = 12(\times 1000/\text{cc}/\text{day})$$

Example – Study of a Population 6

Graph of

$$P(t) = -t^3 + 9\,t^2 - 15\,t + 40,$$

Example – Study of a Population 6

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From the graph, we can guess that the rain fell on the second day of the week with storm runoff polluting the water in the days following