Calculus for the Life Sciences I

Lecture Notes – Applications of the Derivative

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Outline

Applications of the Derivative

Applications of the Derivative

- Sketching Graphs
- Finding Maxima and Minima
- Optimal Values
- Points of Inflection or Steepest parts of a Function

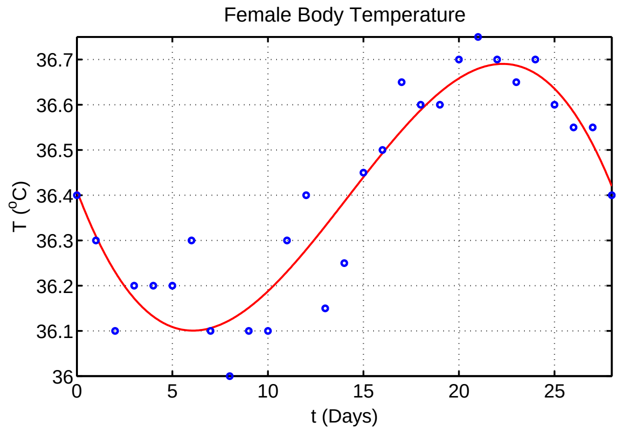
Menstrual Cycle

Body Temperature Fluctuation during the Menstrual Cycle

- Mammals regulate their body temperature in a narrow range to maintain optimal physiological responses
- Variations in body temperature occur during exercise, stress, infection, and other normal situations
- Neurological control of the temperature
- Variations include
  - Circadian rhythms (a few tenths of a degree Celsius)
  - Menstrual cycle - Ovulation often corresponds to the sharpest rise in temperature

Female Basal Body Temperature 1

Female Basal Body Temperature



Application of The Derivative
Examples
Body Temperature Fluctuation
Critical Points
Maxima and Minima
Second Derivative and Concavity
Second Derivative Test
Points of Inflection

Female Basal Body Temperature 2

Female Basal Body Temperature

- The best cubic polynomial fitting the data above is

$$T(t) = -0.0002762\,t^3 + 0.01175\,t^2 - 0.1121\,t + 36.41$$

- Want to find the high and low temperatures
- Determine the time of peak fertility when the temperature is rising most rapidly

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Female Basal Body Temperature 3

Maximum and Minimum for

$$T(t) = -0.0002762\,t^3 + 0.01175\,t^2 - 0.1121\,t + 36.41$$

- The high and low temperatures occur when the curve has slope of zero
- Find the derivative equal to zero
- The derivative of the temperature is

$$T'(t) = -0.0008286\,t^2 + 0.02350\,t - 0.1121$$

- Note that the derivative is a different function from the original function

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Female Basal Body Temperature 4

Maximum and Minimum

- The roots of the derivative (quadratic equation) are

$$t = 6.069 \quad \text{and} \quad 22.29 \text{ days}$$

- Thus,

Minimum at $t = 6.069$ with $T(6.069) = 36.1°C$

Maximum at $t = 22.29$ with $T(22.29) = 36.7°C$

- There is only a $0.6°C$ difference between the high and low basal body temperature during a 28 day menstrual cycle by the approximating function
- The data varied by $0.75°C$

Female Basal Body Temperature 5

Maximum Increase in Temperature

- The maximum increase in temperature is when the derivative is at a maximum
- This is the vertex of the quadratic function, $T'(t)$
- The maximum occurs at the midpoint between the roots of the quadratic equation
- Alternately, the derivative of $T'(t)$ or the second derivative of $T(t)$ equal to zero gives the maximum
- The second derivative is

$$T''(t) = -0.0016572\,t + 0.02350$$

Female Basal Body Temperature 6

Maximum Increase in Temperature

- The second derivative is zero at the **Point of Inflection** at $t = 14.18$ with $T(14.18) = 36.4°C$
- The maximum rate of change in body temperature is

$$T'(14.18) = 0.054°C/\text{day}$$

- This model suggests that the peak fertility occurs on **day 14**, which is consistent with what is known about ovulation

Maxima, Minima, and Critical Points 1

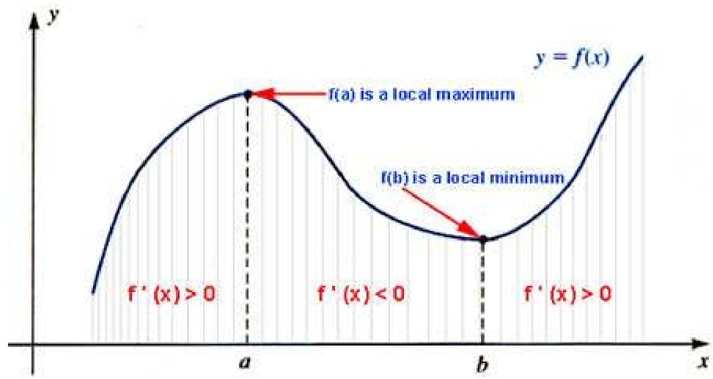
Critical Points, Increasing and Decreasing

The derivative is zero at critical points for the graph of a **smooth function**

Definition: A smooth function $f(x)$ is said to be increasing on an interval (a, b) if $f'(x) > 0$ for all $x \in (a, b)$. Similarly, a smooth function $f(x)$ is said to be decreasing on an interval (a, b) if $f'(x) < 0$ for all $x \in (a, b)$

Maxima, Minima, and Critical Points 2

Maxima, Minima, and Critical Points



Maxima, Minima, and Critical Points 3

Maxima and Minima

- A high point of the graph is where $f(x)$ changes from increasing to decreasing
- A low point on a graph is where $f(x)$ changes from decreasing to increasing
- If $f(x)$ is a smooth function, then either case has the derivative passing through zero

Definition: A smooth function $f(x)$ is said to have a **local** or **relative maximum** at a point c, if $f'(c) = 0$ and $f'(x)$ changes from positive to negative for values of x near c. Similarly, a smooth function $f(x)$ is said to have a **local** or **relative minimum** at a point c, if $f'(c) = 0$ and $f'(x)$ changes from negative to positive for values of x near c.

Maxima, Minima, and Critical Points 4

Critical Points

Definition: If $f(x)$ is a smooth function with $f'(x_c) = 0$, then x_c is said to be a **critical point** of $f(x)$

- Critical points help find the local high and low points on a graph
- Some critical points are neither maxima or minima

Example – Graphing a Polynomial 1

Graphing a Polynomial Consider

$$f(x) = x^3 - 6\,x^2 - 15\,x + 2$$

- Find critical points, maxima, and minima
- Sketch a graph of the function

Example – Graphing a Polynomial 2

With the function

$$f(x) = x^3 - 6\,x^2 - 15\,x + 2$$

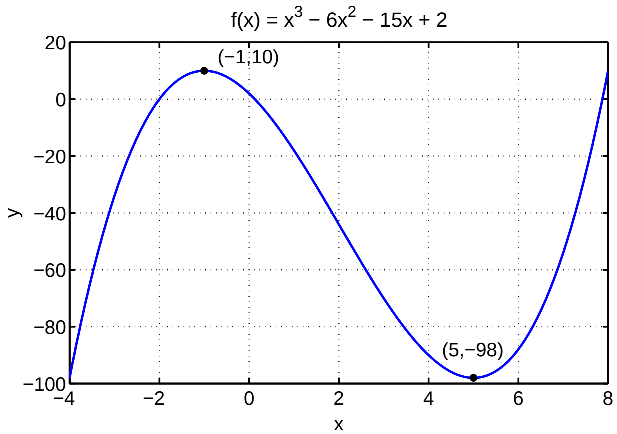
we have the derivative

$$f'(x) = 3\,x^2 - 12\,x - 15 = 3(x+1)(x-5)$$

- The critical points ($f'(x) = 0$) are $x_c = -1$ or 5
- Since $f(-1) = 10$, a **local maximum** occurs at $(-1, 10)$
- Since $f(5) = -98$, a **local minimum** occurs at $(5, -98)$
- The y-intercept is $(0, 2)$

Example – Graphing a Polynomial 3

This gives enough for a reasonable sketch of the graph
Note the x-intercepts are very hard to find



Second Derivative and Concavity

Second Derivative and Concavity

- Since the derivative is itself a function, then if it is differentiable, one can take its derivative to find the **second derivative** often denoted $f''(x)$
- If the first derivative is increasing or the second derivative is positive, then the original function is **concave upward**
- If the first derivative is decreasing or the second derivative is negative, then the original function is **concave downward**
- The second derivative is a measure of the concavity of a function
- For smooth functions, the **maxima** generally occur where the function is **concave downward**, while **minima** occur where the function is **concave upward**

Second Derivative Test

Second Derivative Test: Let $f(x)$ be a smooth function. Suppose that $f'(x_c) = 0$, so x_c is a critical point of f. If $f''(x_c) < 0$, then x_c is a **relative maximum**. If $f''(x_c) > 0$, then x_c is a **relative minimum**.

If $f''(x_c) = 0$, then we get **no information** about the function at the critical point x_c

Example – Graphing a Polynomial 4

Continuing the example $f(x) = x^3 - 6x^2 - 15x + 2$

- The second derivative is $f''(x) = 6x - 12$
- Recall the critical points occurred at $x_c = -1$ and 5
- The second derivative at the critical point $x_c = -1$ gives

$$f''(-1) = -18$$

- The function is concave downward at -1, so this is a relative maximum
- The second derivative at the critical point $x_c = 5$ gives

$$f''(5) = 18$$

- The function is concave upward, so this is a relative minimum

Points of Inflection

Points of Inflection

- When the second derivative is zero, then the function is usually changing from concave upward to concave downward or visa versa
- This is known as a **point of inflection**
- A point of inflection is where the derivative function has a maximum or minimum, so the function is increasing or decreasing most rapidly
- From an applications point of view, if the function is describing a population, then the point of inflection would be where the population is increasing or decreasing most rapidly
- The point of inflection measures when the change of a function is its greatest or smallest

Example – Graphing a Polynomial 5

Continuing the example

$$f(x) = x^3 - 6\,x^2 - 15\,x + 2$$

- The second derivative is

$$f''(x) = 6\,x - 12$$

- The point of inflection is $f''(x) = 0$, which is when $x = 2$
- The point of inflection occurs at $(2, -44)$

Example – Graphing a Cubic Polynomial 1

Graphing a Polynomial: Consider

$$y(x) = 12\,x - x^3$$

- Find the intercepts
- Find critical points, maxima, and minima
- Find the points of inflection
- Sketch the graphs of the function, its derivative, and the second derivative

Skip Example

Example – Graphing a Cubic Polynomial 2

The **y-intercept** for $y(x) = 12\,x - x^3$ is **(0,0)**

The **x-intercepts** satisfy

$$x(12 - x^2) = 0$$

$$x = 0, \pm 2\sqrt{3}$$

Example – Graphing a Cubic Polynomial 3

Maxima and Minima: The **derivative** of $y(x) = 12\,x - x^3$ is

$$y'(x) = 12 - 3\,x^2 = -3(x^2 - 4)$$

The **critical points** satisfy $y'(x_c) = 0$, so

$$x_c = -2 \quad \text{and} \quad x_c = 2$$

- Evaluating the original function at the critical points

$$y(-2) = -16 \quad \text{and} \quad y(2) = 16$$

- The extrema for the function are $(-2, -16)$ and $(2, 16)$
- Clearly, $(-2, -16)$ is a **minimum** and $(2, 16)$ is a **maximum**

Example – Graphing a Cubic Polynomial 4

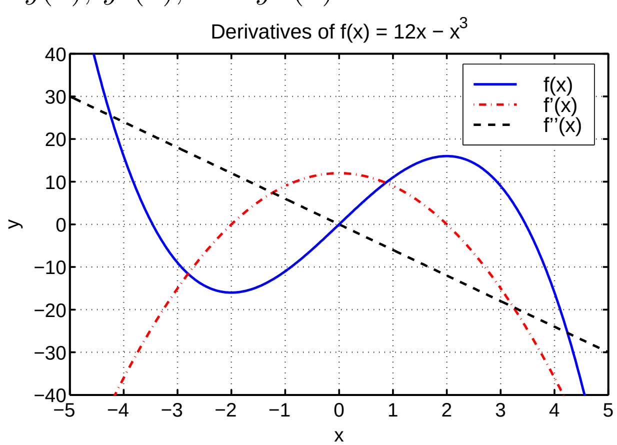
Point of Inflection: The **second derivative** of $y(x) = 12\,x - x^3$ is

$$y''(x) = -6\,x$$

- The **second derivative test** gives
  - $y''(-2) = 12$ is concave upward, so $x_c = -2$ is a minimum
  - $y''(2) = -12$ is concave downward, so $x_c = 2$ is a maximum
- The **Point of Inflection** occurs at $x_p = 0$
- Concavity of the curve changes at $(0, 0)$

Example – Graphing a Cubic Polynomial 5

Graphs of $y(x)$, $y'(x)$, and $y''(x)$



Example – Graphing a Quartic Polynomial 1

Graphing a Polynomial: Consider

$$y(x) = x^4 - 8\,x^2$$

- Find the intercepts
- Find critical points, maxima, and minima
- Find the points of inflection
- Sketch the graphs of the function, its derivative, and the second derivative

Skip Example

Example – Graphing a Quartic Polynomial 2

The **y-intercept** for $y(x) = x^4 - 8\,x^2$ is **(0,0)**

The **x-intercepts** satisfy

$$x^2(x^2 - 8) = 0$$

$$x = 0, \pm 2\sqrt{2}$$

Example – Graphing a Quartic Polynomial 3

Maxima and Minima: The **derivative** of $y(x) = x^4 - 8\,x^2$ is

$$y'(x) = 4\,x^3 - 16\,x = 4\,x(x-2)(x+2)$$

The **critical points** satisfy $y'(x_c) = 0$, so

$$x_c = -2,\ 0,\ 2$$

- Evaluating the original function at the critical points

$$y(-2) = -16, \quad y(0) = 0, \quad \text{and} \quad y(2) = -16$$

- The extrema for the function are $(-2,-16)$, $(0,0)$, and $(2,-16)$
- Clearly, $(-2,-16)$ and $(2,-16)$ are **minima** and $(0,0)$ is a **maximum**

Example – Graphing a Quartic Polynomial 4

The **second derivative** of $y(x) = x^4 - 8\,x^2$ is

$$y''(x) = 12\,x^2 - 16$$

- The **second derivative test** gives
  - $y''(-2) = 32$ is concave upward, so $x_c = -2$ is a minimum
  - $y''(0) = -16$ is concave downward, so $x_c = 0$ is a maximum
  - $y''(2) = 32$ is concave upward, so $x_c = 2$ is a minimum

Example – Graphing a Quartic Polynomial 5

Points of Inflection: Since the **second derivative** of $y(x) = x^4 - 8\,x^2$ is

$$y''(x) = 12\,x^2 - 16 = 4(3\,x^2 - 4)$$

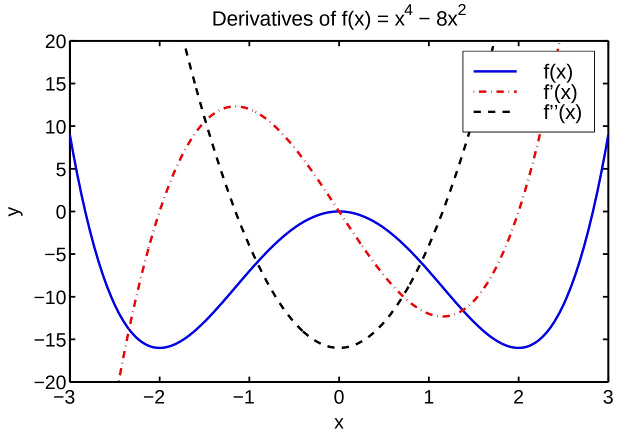
- The **points of inflection** occur when $y''(x) = 0$
- $3\,x_p^2 - 4 = 0$, when

$$x_p = \pm\frac{2}{\sqrt{3}}$$

- Inflection points are $x_p \approx (\pm 1.155, -8.889)$

Example – Graphing a Quartic Polynomial 6

Graphs of $y(x)$, $y'(x)$, and $y''(x)$



Absolute Maxima and Minima 1

Absolute Maxima and Minima: Often we are interested in finding the largest or smallest population over a period of time

Definition: An **absolute minimum** for a function $f(x)$ occurs at a point $x = c$, if $f(c) \leq f(x)$ for all x in the domain of f.

Definition: Similarly, an **absolute maximum** for a function $f(x)$ occurs at a point $x = c$, if $f(c) \geq f(x)$ for all x in the domain of f.

Absolute Maxima and Minima 2

Smooth functions on a closed interval always have an **absolute minimum** and an **absolute maximum**

Theorem: Suppose that $f(x)$ is a continuous, differentiable function on a closed interval $I = [a, b]$, then $f(x)$ achieves its **absolute minimum** (or **maximum**) on I and its minimum (or maximum) occurs either at a point where $f'(x) = 0$ or at one of the endpoints of the interval

This **theorem** says to find the function values at all the critical points and the endpoints of the interval, then this small set of values contains the **absolute minimum** and **absolute maximum**

Example – Study of a Population 1

Study of a Population

- The ocean water is monitored for fecal contamination by counting certain types of bacteria in a sample of seawater
- Over a week where rain occurred early in the week, data were collected on one type of fecal bacteria
- The population of the particular bacteria (in thousand/cc), $P(t)$, were best fit by the cubic polynomial

$$P(t) = -t^3 + 9\,t^2 - 15\,t + 40,$$

where t is in days

Example – Study of a Population 2

Study of a Population

- Find the rate of change in population per day, dP/dt
- What is the rate of change in the population on the third day?
- Find relative and absolute minimum and maximum populations of the bacteria over the time of the surveys
- Determine when the bacterial count is most rapidly increasing
- Sketch a graph of this polynomial fit to the population of bacteria
- When did the rain most likely occur?

Application of The Derivative
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Cubic Polynomial
Quartic Polynomial
Absolute Maxima and Minima
Population Example

Example – Study of a Population 3

Rate of Change in Population: The **derivative** of $P(t) = -t^3 + 9\,t^2 - 15\,t + 40$ is

$$\frac{dP}{dt} = -3\,t^2 + 18\,t - 15$$

Evaluating this on the third day

$$\frac{dP(3)}{dt} = 12(\times 1000/\text{cc}/\text{day})$$

Application of The Derivative
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Population Example

Example – Study of a Population 4

Critical Points: We found

$$\frac{dP}{dt} = -3\,t^2 + 18\,t - 15 = -3(t-1)(t-5)$$

- The **critical points** are
  - $t_c = 1$ and $t_c = 5$
  - **Relative Minimum** at $t_c = 1$ with $P(1) = 33(\times 1000/\text{cc})$
  - **Relative Maximum** at $t_c = 5$ with $P(5) = 65(\times 1000/\text{cc})$
- The endpoint values are $P(0) = 40$ and $P(7) = 33$
- By the theorem above
  - The **absolute maximum** occurs at $t = 5$ with $P(5) = 65$
  - The **absolute minimum** occurs at $t = 1$ and 7 with $P(1) = P(7) = 33$

Application of The Derivative
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Population Example

Example – Study of a Population 5

Point of Inflection: The bacteria is increasing most rapidly when the second derivative is zero

Since $P'(t) = -3\,t^2 + 18\,t - 15$, the second derivative is

$$P''(t) = -6\,t + 18 = -6(t-3)$$

- The population is increasing most rapidly at $t = 3$ with $P(3) = 49(\times 1000/\text{cc})$
- This maximum increase is

$$P'(3) = 12(\times 1000/\text{cc}/\text{day})$$

Example – Study of a Population 6

Graph of

$$P(t) = -t^3 + 9\,t^2 - 15\,t + 40,$$

From the graph, we can guess that the rain fell on the second day of the week with storm runoff polluting the water in the days following