Calculus for the Life Sciences I

Lecture Notes – Allometric Modeling

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Outline

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- The new protease inhibitors have significantly improved the quality of life for those who are HIV positive; however, this has come at a substantial cost to society
- The new drugs are extremely expensive, are difficult to take because of the complex scheduling requirements to be effective, and have many strong side effects (besides not always working for a particular person or strain of the HIV virus)
- In turn, there are a number of people who are now avoiding safe sex practices as they no longer fear the "Death Sentenc" that used to be associated with an HIV infection

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- Below is a table of cumulative cases of AIDS (in thousands)

Year	Cases	Year	Cases
1981	97	1987	29,944
1982	709	1988	83,903
1983	2,698	1989	120,612
1984	6,928	1990	161,711
1985	15,242	1991	206,247
1986	29,944	1992	257,085

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- These techniques are very complicated and often difficult to implement
- **Power Law** or **Allometric Models** are easier

Allometric Models or Power Law Model

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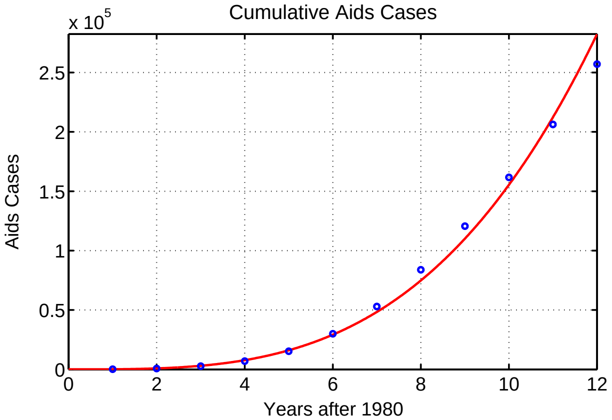
- This model assumes that when $x = 0$, then $y = 0$
- The method fits a straight line to the logarithms of the data

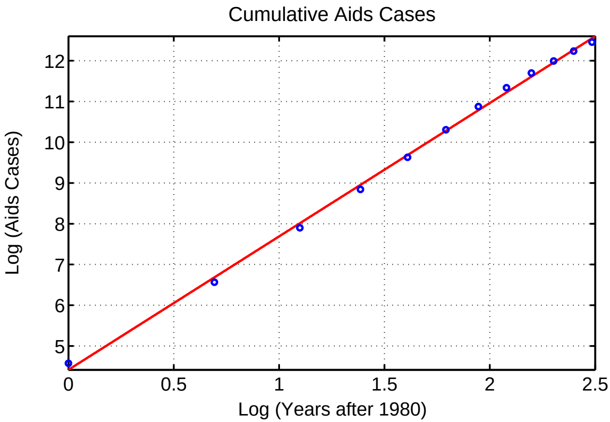
Allometric Model of Cumulative AIDS cases

Graph of the Cumulative AIDS cases



Allometric Model of Cumulative AIDS cases

Graph of the logarithm of the Cumulative AIDS cases against the logarithm of the number of years after 1980



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- The minimum least squares for the log of the data gives $J(A, r) = 0.10$

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- The model predicts 366,990 cases in 1993, which is clearly too high from the given data
- The analysis gives some indication of the rate of growth for this disease
  - Gives a first approximation for improved models
  - Could be applied to expected spread of another disease with similar infectivity as HIV

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- Several Computer Lab problems based on allometric models

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- Allometric Model can give insight into underlying biology of a problem
- Numerous examples satisfy allometric models
- Several Computer Lab problems based on allometric models
- Understanding Allometric models requires logarithms and exponentials

Review of Exponents

Properties of Exponents

1. $a^m a^n = a^{m+n}$,
2. $(a^m)^n = a^{mn}$,
3. $a^{-m} = \dfrac{1}{a^m}$,
4. $\dfrac{a^m}{a^n} = a^{m-n}$,
5. $(ab)^m = a^m b^m$,
6. $a^0 = 1$.

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For solving equations with exponents, the inverse function of the exponent is needed, the **logarithm**

Definition of Logarithm

Definition: Consider the equation:

$$y = a^x$$

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The a in the expression is called the base of the logarithm

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1. $\log(ab) = \log(a) + \log(b)$,
2. $\log(a^m) = m\,\log(a)$,
3. $\log(1/a) = -\log(a)$,
4. $\log(a/b) = \log(a) - \log(b)$,
5. $\log_a(a) = 1$,
6. $\log(1) = 0$.

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- The only property that requires the base of the logarithm for Property 5
- All other properties are independent of which base is used

Base 10 and e

Logarithms base 10 and e

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- The natural logarithm, often denoted $\log$ or $\ln$
  - The natural logarithm is the default for most calculators and programming languages
  - However, Excel defaults to $\log_{10}$
- We will soon see the importance of the natural base e in Calculus

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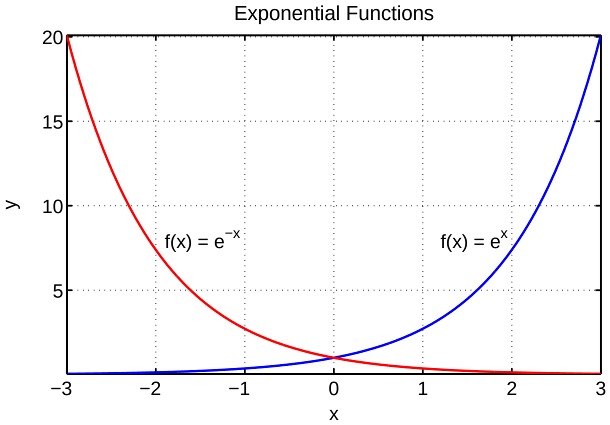
  - It grows very fast for $x > 0$
- The graph of $y = e^{-x}$ has the same y-intercept of 1, but its the mirror reflection through the y-axis of $y = e^x$

Graphing the Exponential Function 2

The graphs of the $y = e^x$ and $y = e^{-x}$



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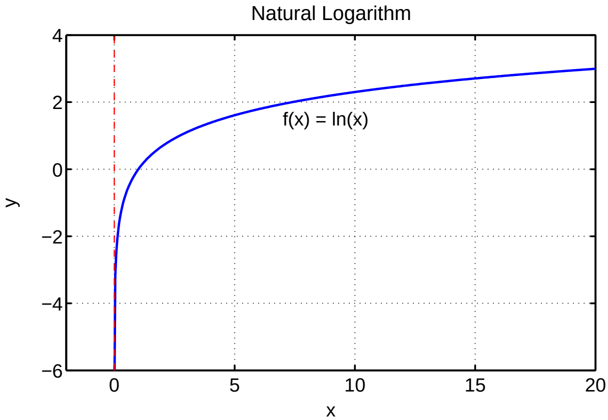
- The domain of $\ln(x)$ is $x > 0$
- The range of $\ln(x)$ is all values of y
- As $y = \ln(x)$ becomes undefined at $x = 0$, there is a vertical asymptote at $x = 0$

Graphing the Logarithm Function 2

The graph of the $y = \ln(x)$



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Consider the exponential function given by

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Find the all intercepts and any horizontal asymptotes and graph this equation

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Solution: Since

$$f(0) = 4 - e^{0} = 4 - 1 = 3,$$

the y-intercept is $(0, 3)$

Example of Graphing the Exponential 2

Solution (cont): Solving $4 - e^{-2x} = 0$ gives

$$e^{-2x} = 4 \qquad \text{or} \qquad e^{2x} = \frac{1}{4}$$

Thus, $2x = \ln(1/4) = -2\ln(2)$ or $x = -\ln(2) \approx -0.6931$

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$f(x)$ tends toward 4

Example of Graphing the Exponential 3

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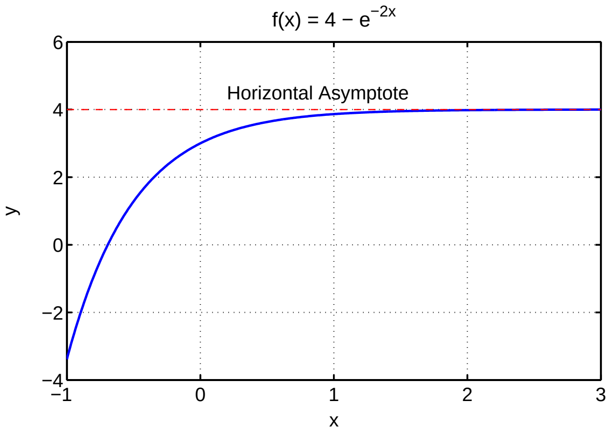

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There is a vertical asymptote at the edge of the domain, where $x = -2$

Example of Graphing the Logarithm 2

Solution (cont): When $x = 0$,

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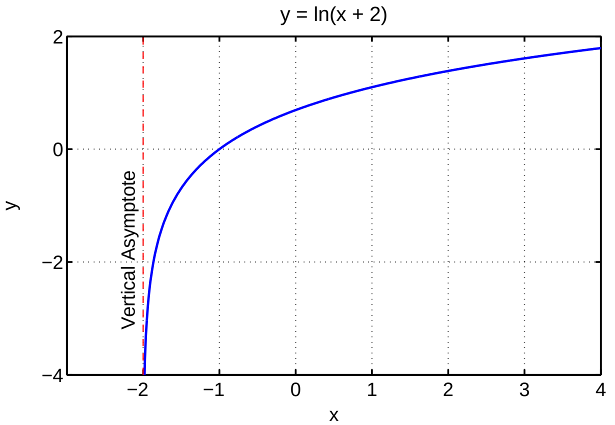
Solving $\ln(x+2) = 0$, gives

$$x + 2 = 1 \qquad \text{or} \qquad x = -1$$

Thus, the x-intercept is $(-1, 0)$

Example of Graphing the Logarithm

The graph of the $y = \ln(x+2)$



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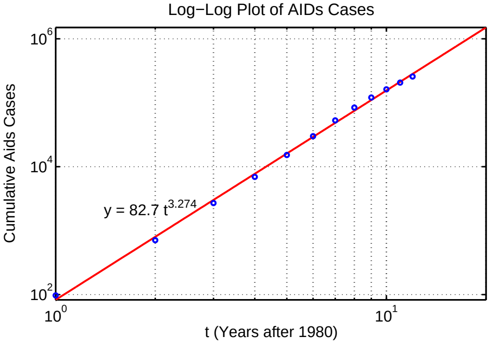
- This is a line with a slope of r and a Y-intercept of $\ln(A)$

Example of Cumulative AIDS cases 1

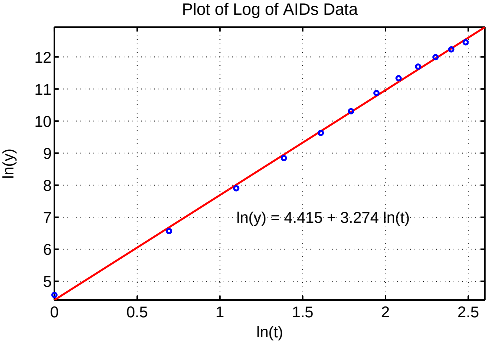
The data for the cumulative AIDS cases is plotted with logarithmic scales



The log-log plot falling on a straight line suggests an **allometric model**

Example of Cumulative AIDS cases

Next we plot the logarithms of the data for cumulative AIDS cases against the logarithms of the time since 1980



The data are flattening for the later years suggesting a diminished rate of increase

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  - It follows $A = 82.70$

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  - Best slope $r = 3.274$
  - Best intercept $a = \ln(A) = 4.415$
  - It follows $A = 82.70$
- The best allometric model is

$$y = 82.70t^{3.274}$$

Example of Weight and Pulse 1

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- Smaller animals have a higher pulse than larger animals

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- Suppose a 17 g mouse has a pulse of 500 beats/min and a 68 kg human has a pulse of 65

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- Suppose a 17 g mouse has a pulse of 500 beats/min and a 68 kg human has a pulse of 65
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- Predict the pulse for a 1.34 kg rabbit

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Example of Weight and Pulse 2

Solution: An allometric model

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As noted above, this is a straight line in $\ln(w)$ and $\ln(P)$

Example of Weight and Pulse

Solution (cont): Create a line through logarithm of data

Animal	Weight(kg)	$\ln(w)$	Pulse(beats/min)	$\ln(P)$
Mouse	0.017	-4.075	500	6.215
Human	68	4.220	65	4.174

Example of Weight and Pulse

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The slope k is

$$k = \frac{4.174 - 6.215}{4.220 + 4.075} = -0.246$$

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The intercept of $\ln(A)$ satisfies:

$$\ln(A) = -k\ln(w_0) + \ln(P_0) = 0.246(4.220) + 4.174 = 5.212$$

Example of Weight and Pulse 4

Solution (cont): The linear model in the logarithm of the data satisfies:

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The **allometric model** is

$$P = 183.5w^{-0.246}$$

Example of Weight and Pulse

Solution (cont): The linear model in the logarithm of the data satisfies:

$$\ln(P) = -0.246\ln(w) + 5.212$$

The **allometric model** is

$$P = 183.5w^{-0.246}$$

If we consider a 1.34 kg rabbit, then the model gives:

$$P = 183.5(1.34)^{-0.246} = 171 \text{ beats/min}$$

Example of Island Biodiversity 1

Example of Island Biodiversity

There are three Pacific islands in a chain. Island A is 15 km^2, Island B is 110 km^2, and Island C is 74 km^2. An extensive biological survey finds 5 species of birds on Island A and 9 species of birds on Island B.

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Assume an allometric model between the number of species, N, on each of these islands and their area, A, of the form

$$N = kA^x$$

Use the data from Islands A and B to determine constants k and x

Example of Island Biodiversity 2

From the model predict the number of bird species on Island C

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From the model predict the number of bird species on Island C
Also, determine how large an island would be required to support 20 species of birds near this chain of islands

Example of Island Biodiversity

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Solution: The allometric model can be rewritten as the linear model of the logarithm of the data

$$\ln(N) = \ln(k) + x \ln(A)$$

Example of Island Biodiversity

Solution (cont): Create a line through logarithm of data

Island	Area	$\ln(A)$	Species	$\ln(N)$
A	15	2.708	5	1.609
B	110	4.700	9	2.197

Example of Island Biodiversity 3

Solution (cont): Create a line through logarithm of data

Island	Area	$\ln(A)$	Species	$\ln(N)$
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$$x = \frac{\ln(9) - \ln(5)}{\ln(110) - \ln(15)} \approx 0.295$$

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The intercept of $\ln(k)$ satisfies:

$$\ln(k) = \ln(5) - x\ln(15) \approx 0.811 \quad \text{or} \quad k \approx 2.25$$

Example of Island Biodiversity 4

Solution (cont): The linear model in the logarithm of the data satisfies:

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Example of Island Biodiversity 4

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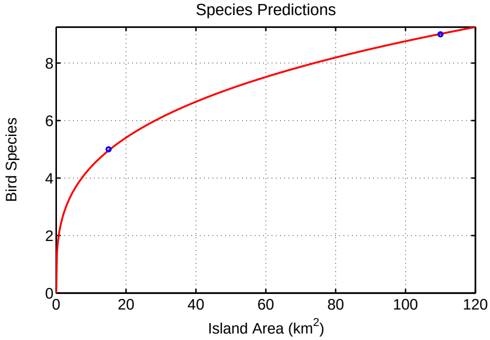
The model predicts that Island C has

$$N = 2.25(74)^{0.295} \approx 8.01 \text{ species}$$

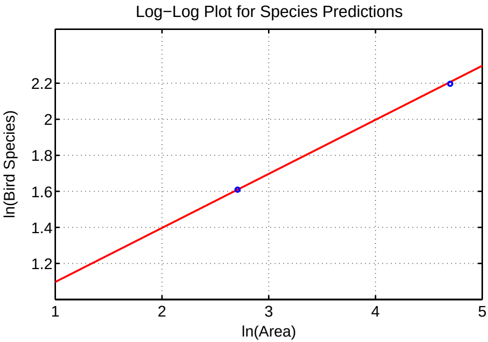
Example of Island Biodiversity 5

Below is a graph of the allometric model showing the area and the species predictions



Example of Island Biodiversity

Below is a graph of $\ln(A)$ and the $\ln(N)$, showing the linearity of the logarithmic plot



Example of Island Biodiversity

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Example of Island Biodiversity 7

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The model predicts that the area of the island needs to be $A = 1646\text{ km}^2$