Calculus for the Life Sciences I

Lecture Notes – Allometric Modeling

Joseph M. Mahaffy,
⟨mahaffy@math.sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://www-rohan.sdsu.edu/∼jmahaffy

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Outline

Cumulative AIDS cases

Cumulative AIDS cases

- AIDS has had a significant impact on both personal behavior and public policy
- The new protease inhibitors have significantly improved the quality of life for those who are HIV positive; however, this has come at a substantial cost to society
- The new drugs are extremely expensive, are difficult to take because of the complex scheduling requirements to be effective, and have many strong side effects (besides not always working for a particular person or strain of the HIV virus)
- In turn, there are a number of people who are now avoiding safe sex practices as they no longer fear the "Death Sentenc" that used to be associated with an HIV infection

Predicting AIDS Cases

Predicting AIDS Cases

- Society needs to know the extent of this disease from both an economic and sociological perspective
- Want to know what is the expected case load in the future
- This is an extremely complex modeling problem
- Below is a table of cumulative cases of AIDS (in thousands)

Year	Cases	Year	Cases
1981	97	1987	29,944
1982	709	1988	83,903
1983	2,698	1989	120,612
1984	6,928	1990	161,711
1985	15,242	1991	206,247
1986	29,944	1992	257,085

Modeling Data

Modeling the Data

- The data is clearly not linear
- There are general methods for finding the least squares best fit to nonlinear data
- These techniques are very complicated and often difficult to implement
- **Power Law** or **Allometric Models** are easier

Allometric Models or Power Law Model

Allometric Models

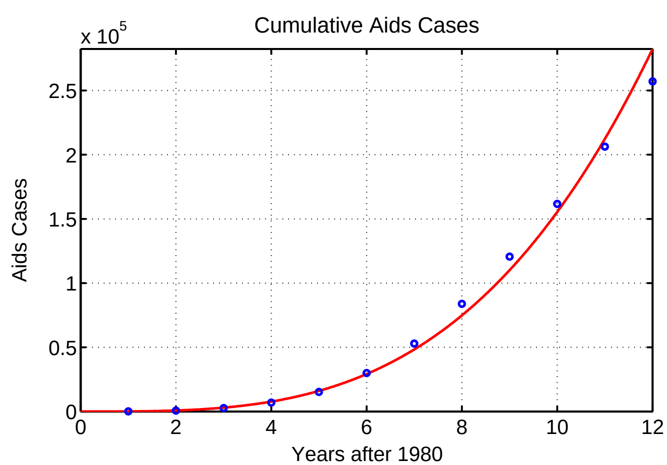
- Allometric models assume a relationship between two sets of data, x and y, that satisfy a power law of the form

$$y = Ax^r$$

- A and r are parameters that are chosen to best fit the data in some sense
- This model assumes that when $x = 0$, then $y = 0$
- The method fits a straight line to the logarithms of the data

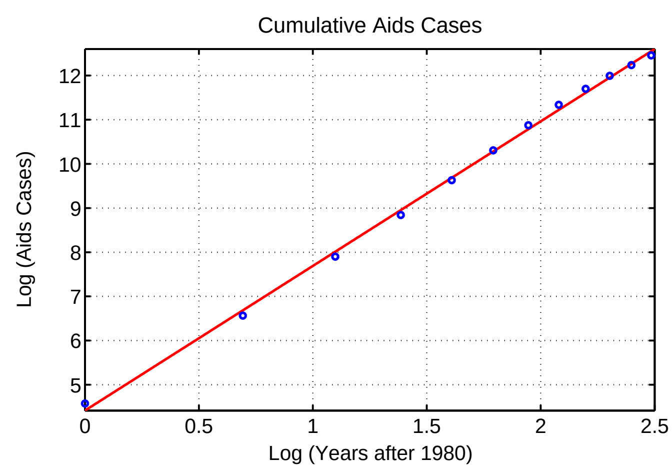
Allometric Model of Cumulative AIDS cases

Graph of the Cumulative AIDS cases



Allometric Model of Cumulative AIDS cases

Graph of the logarithm of the Cumulative AIDS cases against the logarithm of the number of years after 1980



Allometric Models of Cumulative AIDS cases

Allometric Model for Cumulative AIDS cases

- The best slope is $r = 3.27$
- The best intercept is $\ln(A) = 4.42$ with $A = 82.7$
- This gives the best fit power law for this model as

$$y = 82.7x^{3.27}$$

- The minimum least squares for the log of the data gives $J(A, r) = 0.10$

Cumulative AIDS cases Model

Allometric Model for Cumulative AIDS cases

- The graph of the power law provides a reasonable fit to the data
- The fit is weakest at the end where we'd like to use the model to predict the cumulative AIDS cases for the next year
- The model predicts 366,990 cases in 1993, which is clearly too high from the given data
- The analysis gives some indication of the rate of growth for this disease
  - Gives a first approximation for improved models
  - Could be applied to expected spread of another disease with similar infectivity as HIV

Allometric Model/Power Law

Allometric Model/Power Law

- When the logarithm of the data lie on a line, then a **Allometric Model** is appropriate
- Allometric Model can give insight into underlying biology of a problem
- Numerous examples satisfy allometric models
- Several Computer Lab problems based on allometric models
- Understanding Allometric models requires logarithms and exponentials

Review of Exponents

Properties of Exponents

1. $a^m a^n = a^{m+n}$,
2. $(a^m)^n = a^{mn}$,
3. $a^{-m} = \dfrac{1}{a^m}$,
4. $\dfrac{a^m}{a^n} = a^{m-n}$,
5. $(ab)^m = a^m b^m$,
6. $a^0 = 1$.

For solving equations with exponents, the inverse function of the exponent is needed, the **logarithm**

Definition of Logarithm

Definition: Consider the equation:

$$y = a^x$$

The inverse equation that solves for x is given by

$$x = \log_a y$$

The a in the expression is called the base of the logarithm

Review of Logarithms

Properties of Logarithms

1. $\log(ab) = \log(a) + \log(b)$,
2. $\log(a^m) = m\log(a)$,
3. $\log(1/a) = -\log(a)$,
4. $\log(a/b) = \log(a) - \log(b)$,
5. $\log_a(a) = 1$,
6. $\log(1) = 0$.

- The only property that requires the base of the logarithm for Property 5
- All other properties are independent of which base is used

Base 10 and e

Logarithms base 10 and e

- The two most common logarithms that are used are $\log_{10}$ and $\log_e$
- The natural logarithm, often denoted log or ln
  - The natural logarithm is the default for most calculators and programming languages
  - However, Excel defaults to $\log_{10}$
- We will soon see the importance of the natural base e in Calculus

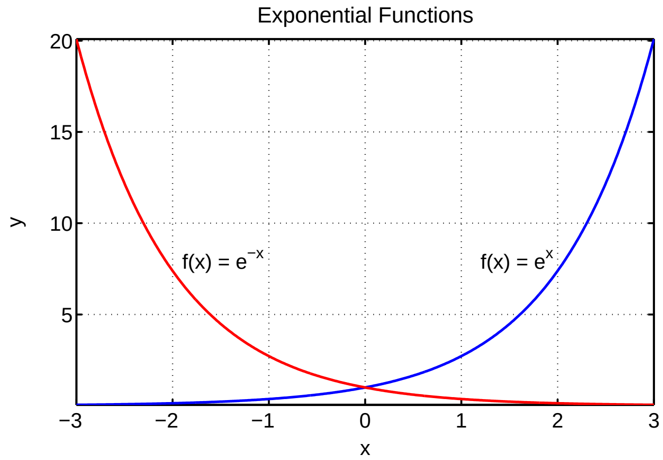
Graphing the Exponential Function 1

Graphing the Exponential Function

- For graphing purposes, e is an irrational number between 2 and 3, more precisely, $e = 2.71828...$
- The domain of e^x is all of x
  - e^x becomes extremely small very fast for $x < 0$ (a horizontal asymptote of $y = 0$)
  - It grows very fast for $x > 0$
- The graph of $y = e^{-x}$ has the same y-intercept of 1, but its the mirror reflection through the y-axis of $y = e^x$

Graphing the Exponential Function 2

The graphs of the $y = e^x$ and $y = e^{-x}$



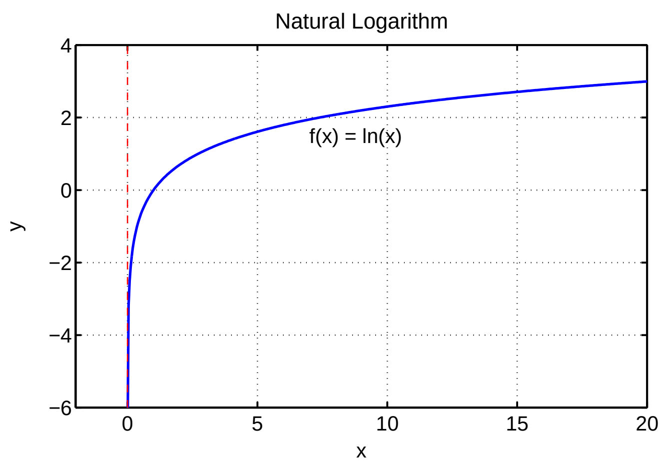
Graphing the Logarithm Function 1

Graphing the Logarithm Function

- Since $\ln(x)$ is the inverse function of e^x, the graph of this function mirrors the graph of e^x through the line $y = x$
- The domain of $\ln(x)$ is $x > 0$
- The range of $\ln(x)$ is all values of y
- As $y = \ln(x)$ becomes undefined at $x = 0$, there is a vertical asymptote at $x = 0$

Graphing the Logarithm Function 2

The graph of the $y = \ln(x)$



Example of Graphing the Exponential 1

Consider the exponential function given by

$$f(x) = 4 - e^{-2x}$$

Find the all intercepts and any horizontal asymptotes and graph this equation

Skip Example

Solution: Since

$$f(0) = 4 - e^0 = 4 - 1 = 3,$$

the y-intercept is $(0, 3)$

Example of Graphing the Exponential 2

Solution (cont): Solving $4 - e^{-2x} = 0$ gives

$$e^{-2x} = 4 \qquad \text{or} \qquad e^{2x} = \frac{1}{4}$$

Thus, $2x = \ln(1/4) = -2\ln(2)$ or $x = -\ln(2) \approx -0.6931$

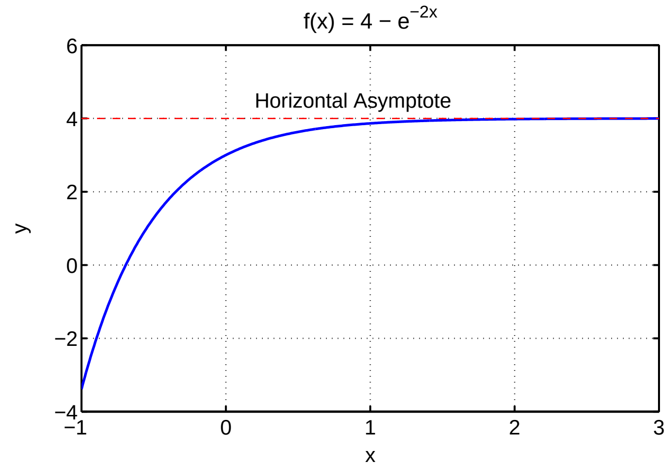
The x-intercept is $(-0.6931, 0)$

For large values of x, e^{-2x} is very close to zero, so there is a horizontal asymptote for large positive x

$f(x)$ tends toward 4

Example of Graphing the Exponential 3

The graph of the $y = 4 - e^{-2x}$



Example of Graphing the Logarithm 1

Consider the exponential function given by

$$f(x) = \ln(x + 2)$$

Find the all intercepts and any vertical asymptotes and graph this equation

Skip Example

Solution: The **domain** of $f(x)$ is $x > -2$

There is a vertical asymptote at the edge of the domain, where $x = -2$

Example of Graphing the Logarithm 2

Solution (cont): When $x = 0$,

$$f(0) = \ln(2) \approx 0.6931$$

Thus, the y-intercept is $(0, 0.6931)$

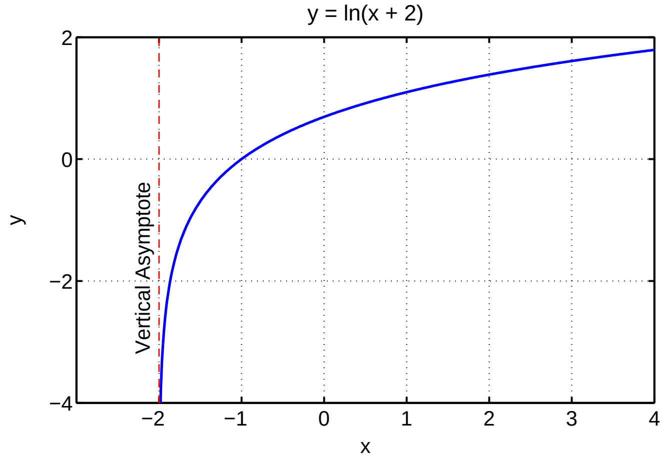
Solving $\ln(x + 2) = 0$, gives

$$x + 2 = 1 \qquad \text{or} \qquad x = -1$$

Thus, the x-intercept is $(-1, 0)$

Example of Graphing the Logarithm 3

The graph of the $y = \ln(x+2)$



Allometric Models

The **Allometric Model** for two sets of data, x and y satisfies a power law

$$y = Ax^r$$

- The parameters A and r are chosen to best fit the data
- By properties of logarithms

$$\ln(y) = \ln(Ax^r) = \ln(A) + r\ln(x)$$

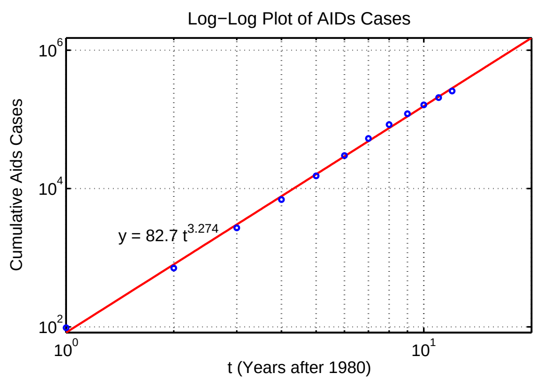
- Let $X = \ln(x)$, $Y = \ln(y)$, and $a = \ln(A)$, then

$$Y = a + rX$$

- This is a line with a slope of r and a Y-intercept of $\ln(A)$

Example of Cumulative AIDS cases 1

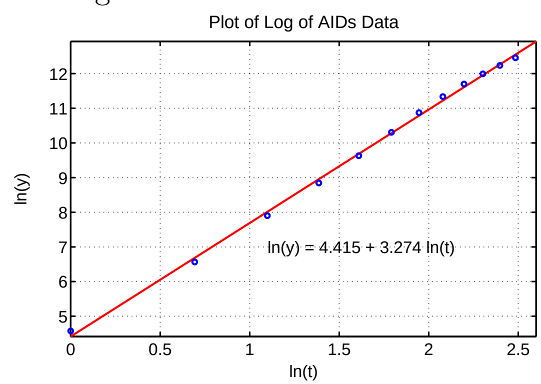
The data for the cumulative AIDS cases is plotted with logarithmic scales



The log-log plot falling on a straight line suggests an **allometric model**

Example of Cumulative AIDS cases 2

Next we plot the logarithms of the data for cumulative AIDS cases against the logarithms of the time since 1980



The data are flattening for the later years suggesting a diminished rate of increase

Example of Cumulative AIDS cases 3

Allometric Model for Cumulative AIDS cases

- The least squares best fit of the straight line to the logarithms of the data
  - Best slope $r = 3.274$
  - Best intercept $a = \ln(A) = 4.415$
  - It follows $A = 82.70$
- The best allometric model is

$$y = 82.70t^{3.274}$$

Example of Weight and Pulse 1

Example of Weight and Pulse

- Smaller animals have a higher pulse than larger animals
- Suppose a 17 g mouse has a pulse of 500 beats/min and a 68 kg human has a pulse of 65
- Create an allometric model (other data support this form)
- Predict the pulse for a 1.34 kg rabbit

Skip Example

Example of Weight and Pulse 2

Solution: An allometric model

$$P = Aw^k$$

Logarithms give

$$\ln(P) = \ln(A) + k\ln(w)$$

As noted above, this is a straight line in $\ln(w)$ and $\ln(P)$

Example of Weight and Pulse 3

Solution (cont): Create a line through logarithm of data

Animal	Weight(kg)	$\ln(w)$	Pulse(beats/min)	$\ln(P)$
Mouse	0.017	−4.075	500	6.215
Human	68	4.220	65	4.174

The slope k is

$$k = \frac{4.174 - 6.215}{4.220 + 4.075} = -0.246$$

The intercept of $\ln(A)$ satisfies:

$$\ln(A) = -k\ln(w_0) + \ln(P_0) = 0.246(4.220) + 4.174 = 5.212$$

Example of Weight and Pulse 4

Solution (cont): The linear model in the logarithm of the data satisfies:

$$\ln(P) = -0.246\ln(w) + 5.212$$

The **allometric model** is

$$P = 183.5w^{-0.246}$$

If we consider a 1.34 kg rabbit, then the model gives:

$$P = 183.5(1.34)^{-0.246} = 171 \text{ beats/min}$$

Example of Island Biodiversity 1

Example of Island Biodiversity

There are three Pacific islands in a chain. Island A is 15 km^2, Island B is 110 km^2, and Island C is 74 km^2. An extensive biological survey finds 5 species of birds on Island A and 9 species of birds on Island B.

Skip Example

Assume an allometric model between the number of species, N, on each of these islands and their area, A, of the form

$$N = kA^x$$

Use the data from Islands A and B to determine constants k and x

Example of Island Biodiversity 2

From the model predict the number of bird species on Island C
Also, determine how large an island would be required to support 20 species of birds near this chain of islands

Solution: The allometric model can be rewritten as the linear model of the logarithm of the data

$$\ln(N) = \ln(k) + x\ln(A)$$

Example of Island Biodiversity 3

Solution (cont): Create a line through logarithm of data

Island	Area	$\ln(A)$	Species	$\ln(N)$
A	15	2.708	5	1.609
B	110	4.700	9	2.197

The slope x satisfies

$$x = \frac{\ln(9) - \ln(5)}{\ln(110) - \ln(15)} \approx 0.295$$

The intercept of $\ln(k)$ satisfies:

$$\ln(k) = \ln(5) - x\ln(15) \approx 0.811 \quad \text{or} \quad k \approx 2.25$$

Example of Island Biodiversity

Solution (cont): The linear model in the logarithm of the data satisfies:

$$\ln(N) = 0.811 + 0.295\ln(A)$$

The **allometric model** is

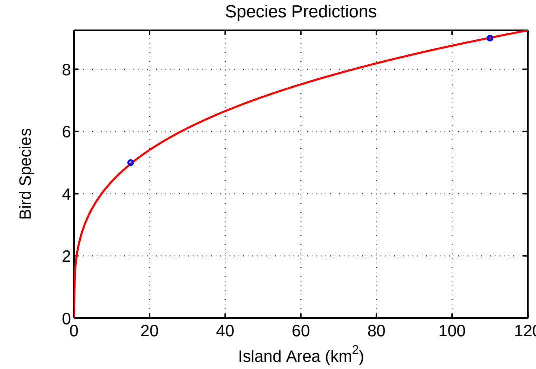
$$N = 2.25A^{0.295}$$

The model predicts that Island C has

$$N = 2.25(74)^{0.295} \approx 8.01 \text{ species}$$

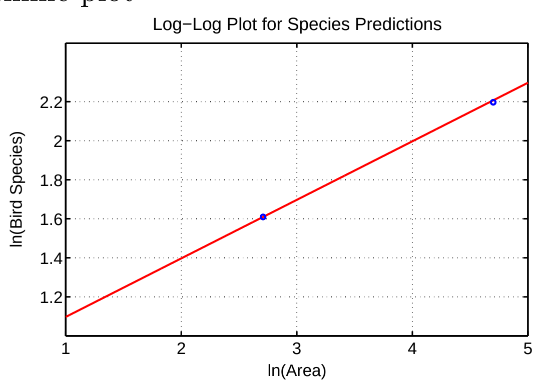
Example of Island Biodiversity

Below is a graph of the allometric model showing the area and the species predictions



Example of Island Biodiversity

Below is a graph of $\ln(A)$ and the $\ln(N)$, showing the linearity of the logarithmic plot



Example of Island Biodiversity

Finally, we want to predict the size of an island necessary to support 20 species

We solve the equation

$$20 = 2.25A^{0.295}$$

From properties of logarithms

$$\ln(20) = \ln(2.25) + 0.295\ln(A)$$

$$\ln(A) = \frac{1}{0.295}(\ln(20) - \ln(2.25) = 7.406$$

The model predicts that the area of the island needs to be $A = 1646 \text{ km}^2$