Spring 2008 Math 121a Name ______________________________

Make sure there are **8** pages on this test. You should **read** all questions first.
Give your answers to **at least 4 significant figures** whenever possible.

1. (20pts) Differentiate the following functions (you **don't** have to simplify):

a. $f(t) = 4\ln(t^3) - \dfrac{3}{t^2}$.

$f'(t) =$ ______________________________

b. $g(x) = (8x + e^{2x^2 - x})^4 - 2$.

$g'(x) =$ ______________________________

c. $h(t) = t^3 \ln(t^2 + 5)$.

$h'(t) =$ ______________________________

d. $k(x) = \dfrac{\sqrt{x}}{x^2 + e^{2x}}$.

$k'(x) =$ ______________________________

2. (30pts) Sketch the graph of the following functions. Give the x and y-intercepts, and any asymptotes. Find the derivative and its critical point(s) (including the x and y values). Indicate whether it is a local maxima or minima. For function. (If the function does not have a particular asymptote, extrema or x or y-intercept, indicate "NONE").

a. $y = x + \frac{9}{x}$. **Graph of $y(x)$:**

x-intercept(s) ____________________

y-intercept __________

Vertical asymptote(s) ____________________

Horizontal asymptote(s) ____________________

$y'(x) =$ __________________________________

$x_{max} =$__________ $y(x_{max}) =$__________

$x_{min} =$__________ $y(x_{min}) =$__________

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b. $y = \frac{3x^2}{x^2 - 9}$. **Graph of $y(x)$:**

x-intercept(s) __________

y-intercept is __________

Vertical Asymptote(s) ____________________

Horizontal Asymptote(s) ____________________

$y'(x) =$ __________________________________

$x_{max} =$__________ $y(x_{max}) =$__________

$x_{min} =$__________ $y(x_{min}) =$__________

3. (25pts) A number of trees from the Alleghany National Forest were measured to provide information about the quantity of wood in the forest. One tree had had a diameter (d) of 8.2 inches and produced a volume (V) of 10.7 board feet. Another tree measured 14.5 inches in diameter and had a volume of 36.3 board feet. (Give all answers to **4 significant figures**.)

a. A linear model is given by:

$$V = md + b,$$

for some constants m and b. Find the constants m and b and sketch a graph of this model. Use this model to predict the number of board feet in a tree that has a diameter of 13 inches. Also, find the diameter of a tree that produces 40 board feet.

$m =$ __________

$b =$ __________

If $d = 13$ in, then $V =$ __________ board ft,

If $V = 40$ board ft, then $d =$ __________ in.

b. If the relationship between the diameter and volume of wood for the trees in this forest satisfies a power law of the form

$$V = kd^a,$$

Find the constants k and a and sketch a graph of this model. Use this model to predict the number of board feet in a tree that has a diameter of 13 inches. Also, find the diameter of a tree that produces 40 board feet.

$a =$ __________

$k =$ __________

If $d = 13$ in, then $V =$ __________ board ft,

If $V = 40$ board ft, then $d =$ __________ in.

c. Which model provides the better estimate?

Circle one: **Linear Model** **Allometric Model**

4. (25pts) **a.** The population of France in 1950 was about 41.8 million, while in 1970, it was about 50.8 million. Assume that the population is growing according to the discrete Malthusian growth equation

$$P_{n+1} = (1 + r)P_n, \quad \text{with} \quad P_0 = 41.8,$$

where P_0 is the population in 1950 and n is in decades. Use the population in 1970 (P_2) to find the value of r (to 4 significant figures). If France's population continued to grow in this manner, find the time for it to double.

$r =$ __________ Doubling time $=$ __________

b. Estimate the population in 2000 based on this model. Given that the population in 2000 was 59.4 million, find the percent error between the actual and predicted values.

$P_5 =$ __________ % error $=$ __________

c. A better model fitting the census data for France is a Logistic growth model given by

$$P_{n+1} = P_n + G(P_n) = P_n + 0.28P_n - 0.00416P_n^2,$$

where again n is in decades after 1950. This form of the Logistic growth model has the growth function:

$$G(P) = 0.28P - 0.00416P^2.$$

Equilibria occur when $G(P) = 0$. Find the equilibrium populations of France predicted by this model. Compute the derivative of the growth function, $G'(P)$, then determine the population at which this growth function is at a maximum. What is the maximum rate of growth in population for France? Sketch the graph of $G(P)$.

Graph of $G(P)$:

All Equilibria $=$ ____________________

$G'(P) =$ ____________________________________

$P_{max} =$ __________ $G(P_{max}) =$ __________

5. (25pts) An invasive species of insect enters an ecological study area. The initial survey of the region finds a population of 150. A week later the population is found to be 200 insects in the study area.

a. The first group of investigators, assuming that these insects descended from one invading insect, applied a Malthusian growth model:

$$P_{n+1} = (1+r)P_n, \qquad P_0 = 150,$$

where r is the growth rate and n is in weeks. With the information that $P_1 = 200$, solve this growth model for this invasive species, finding the growth rate r from these data. How long does it take for this insect population to double according to this model? What does this model predict is the population of insects after 4 weeks?

$r =$ __________

Doubling time = __________ $P_4 =$ __________

b. A second group of ecologists examines the insects more closely, and their molecular studies show that the insects come from a more diverse population. This suggests to these scientists that insects continue to enter this population from outside, so a better model is the discrete Malthusian growth model with immigration:

$$P_{n+1} = (1+r)P_n + \mu, \qquad P_0 = 150 \quad \text{and} \quad P_1 = 200.$$

They make another survey and find $P_2 = 263$. Use these data to find the growth rate r and the immigration rate μ. Simulate the model to predict the populations P_3 and P_4.

$r =$ __________ and $\mu =$ __________

$P_3 =$ __________ and $P_4 =$ __________

c. A survey after 4 weeks finds a population of 450 insects. Find the percent error from each of the models above. (Assume that the survey value is the best value for computing your error.)

% error (Model Part **a.**) = __________ % error (Model Part **b.**) = __________

6. (25pts) In lab we saw the experimental fit of O_2 consumption (in μl/hr) after a blood meal by the beetle *Triatoma phyllosoma*. Below is a cubic polynomial fit to measurements for a different individual "kissing bug,"

$$Y(t) = \frac{1}{3}\,t^3 - 6\,t^2 + 20\,t + 120,$$

where t is in hours, for $0 \le t \le 12$.

a. Find the rate of change in O_2 consumption per hour, $\frac{dY}{dt}$. What is the rate of change in the O_2 consumption at $t = 4$? Also, compute $Y''(t)$. When is the rate of change in the O_2 consumption per hour decreasing the most and what is that maximum rate of increase?

$Y'(t) =$ ______________________________ $\qquad Y'(4) =$ __________

$Y''(t) =$ ________________________

Rate of maximum decrease at $t_{dec} =$ __________ $\qquad Y'(t_{dec}) =$ __________

b. Use the derivative to find when the minimum and maximum O_2 consumption for this beetle occurs during the experiment. Give the O_2 consumption at those times.

$t_{max} =$ ________________ $\qquad Y(t_{max}) =$ ________________

$t_{min} =$ ________________ $\qquad Y(t_{min}) =$ ________________

c. Sketch a graph of this polynomial fit to the O_2 consumption. Show clearly the maximum and minimum O_2 consumption on your graph and include the O_2 consumption at the beginning of the study ($t = 0$) and at the end ($t = 12$).

$Y(0) =$ __________ $\qquad Y(12) =$ __________

Graph of $Y(t)$:

7. (25pts) Gompertz developed a model for the growth of a tumor. A particular type of tumor growing according to Gompertz's model satisfies the growth law,

$$G(N) = N(4.9 - 0.42\ln(N)) \quad \text{(cells/day)},$$

where N is the number of tumor cells and the time units are days. Note that this model is not defined for $N = 0$, and has been shown to work poorly for very low populations of tumor cells.

a. Find all equilibria for this Gompertz model by solving $G(N) = 0$.

$N_e =$____________

b. Compute the derivative of $G(N)$ and find the maximum of the function (both N and $G(N)$ values).

$G'(N) =$____________________________________

$N_{max} =$__________ $G(N_{max}) =$__________

c. Find the value for the growth function for $N = 60,000$ cells. Also find the rate of the growth function of the tumor for this size. Describe whether the tumor is growing or decreasing according to your results, and whether the velocity of this growth/decrease is increasing or decreasing.

$G(60,000) =$__________ $G'(60,000) =$__________

For $N = 60,000$:

The tumor is (circle one:) growing shrinking

The rate of growth is (circle one:) increasing decreasing.

Sketch of GRAPH

8. (25pts) Hassell's model is used to study population of insects. Let P_n be the population of a species of beetles in weeks n and suppose that Hassell's model is given by

$$P_{n+1} = H(P_n) = \frac{20P_n}{(1 + 0.004P_n)^4}.$$

a. Sketch a graph of $H(P)$ with the identity function for $P \geq 0$, showing the intercepts and any horizontal asymptotes. Find $H'(P)$, then determine the maximum of this function (both P and $H(P)$ values).

P-intercept ______ H-intercept ______

Horizontal Asymptote $H =$ ______

$H'(P) =$ ____

$P_{max} =$ _________ $H(P_{max}) =$ _________

Graph of P_n vs. P_{n+1}:

b. Find the two equilibria P_{1e} and P_{2e} (such that $P_{1e} < P_{2e}$). And give the value of the derivative of $H(P)$ at those equilibria populations.

$P_{1e} =$ _______ $H'(P_{1e}) =$ _______

$P_{2e} =$ _______ $H'(P_{2e}) =$ _______