1. Differentiate the following:

a. $f(x) = 6x^3 + \dfrac{2}{x^2} - e^{2x}(x^2 - 9)$,

b. $g(x) = 2e^{-3x} + \ln(x^2) - 5$,

c. $h(x) = 2x^6 \ln(x) - e^{x^2+4x} + \frac{e^{-4x}}{2}$,

d. $k(t) = \dfrac{1}{4}t^2 - \dfrac{4}{\sqrt{t}} + \dfrac{2 + e^{2t}}{t^2 - 3}$,

e. $p(w) = \dfrac{2\ln(w) + w}{w^3 - 8} - w^{2/5} + \dfrac{1}{w^3}e^{-w}$,

f. $q(z) = Az\ln(z^2) - B\sqrt{z} + \dfrac{C}{z^3}$.

g. $r(x) = e^{2x}(x^3 - 5x + 7)^4 - \dfrac{7x}{\sqrt{x^2 + 2x + 5}}$,

h. $F(y) = (3y^2 - 4y + 6)^5 + \ln(2y + 9)$.

2. For each of the following functions, give the domain. Find all x and y-intercepts and any asymptotes, if they exist. Find the derivative of the functions, then determine any maxima or minima. Give both the x and y values. Write the second derivative and find any points of inflection. Finally, sketch the graph of the function.

a. $y = 4xe^{-0.02x}$,

b. $y = (x + 3)\ln(x + 3)$,

c. $y = (x - 4)e^{2x}$,

d. $y = \dfrac{10(x - 2)}{(1 + 0.5x)^3}$,

e. $y = \dfrac{2\,e^{2x}}{x - 1}$,

f. $y = \dfrac{4x^2}{x + 3}$

3. Plants require nutrients to grow, but an excess of any one nutrient can be toxic. Suppose that it is found that the rate of growth of a plant (in mm/day) satisfies the equation:

$$R(n) = \frac{10n}{1 + n^2},$$

where n is the concentration of the essential nutrient (in mg/l).

a. Find the rate of change in growth rate as a function of nutrient application, $\dfrac{dR}{dn}$. What is the rate of change in the growth rate when $n = 2$?

b. Use the derivative to any relative minima or maxima for $R(n)$ for $n \geq 0$. Give both the n and R values.

c. Find the n and R-intercepts and any asymptotes. Sketch a graph of this function for $n \geq 0$.

4. Some hormones have a strong effect on mood, so finding a delivery device that delivers a hormone at a more constant level over a longer period of time is important for hormone therapy. Suppose that a drug company finds a polymer that can be implanted to deliver a hormone, $h(t)$, which is experimentally found to satisfy

$$h(t) = 40\left(e^{-0.005t} - e^{-0.15t}\right),$$

where h is in nanograms per deciliter of blood (ng/dl) and t is in days.

a. Find the maximum concentration of this hormone in the body and when this occurs.

b. Determine all intercepts and asymptotes, then graph $h(t)$ for $0 \le t \le 150$. Use the graph to approximate how long the hormone level remains above 20 ng/dl.

5. The distribution of seeds from around a plant satisfies an exponential distribution. It is more likely to find seeds close to a plant than it is far away. Suppose that experimental measurements fitting the seed distribution radially from the plant satisfies

$$P(r) = 0.04r\, e^{-0.2r},$$

where P is the probability of finding a seed r meters from the plant. Find the distance r and the probability $P(r)$ at which a seed is most likely to land. That is find the maximum probability from the function above. Sketch a graph of $P(r)$ showing any intercepts, asymptotes, and local extrema.

6. A tumor growing according to Gompertz's model satisfies the growth law

$$G(N) = N(0.8 - 0.04\ln(N)) \text{ (cells/day)},$$

where N is the number of tumor cells and the time units are days.

a. Find the equilibrium number of tumor cells by solving when $G(N) = 0$.

b. Compute $G'(N)$ and determine the population at which the maximum rate of growth of the tumor is occurring. What is the maximum growth rate (in cells/day)?

c. Find the value for the growth function for $N = 2 \times 10^8$. Also, find the rate of the growth function of the tumor for this size. Describe whether the tumor is growing or decreasing according to your results, and whether the velocity of this growth/decrease is increasing or decreasing.

7. a. After ingestion of a sugar-rich meal, the concentration of glucose rises very rapidly in the blood, then through insulin, the glucose is converted to glycogen for later use as energy. An approximation for the concentration of glucose in the body after this meal is given by

$$g(t) = 70 + 90e^{-0.7t},$$

where t is in hours and g is in mg/100 ml of blood. Find how long it takes for the concentration of glucose in the blood to reach 90 mg/100 ml of blood. Sketch a graph for the concentration of glucose in the blood.

b. Find the rate of change of glucose per hour ($\frac{dg}{dt}$) for any time t, then evaluate this rate of change at $t = 1$.

c. The release of insulin responds to this rise in glucose. Suppose that the level of insulin satisfies the function

$$i(t) = 10(e^{-0.4t} - e^{-0.5t}).$$

Find when the insulin reaches its maximum concentration and determine what its maximum concentration is. Sketch a graph of the insulin concentration.

d. Find the rate of change of insulin per hour ($\frac{di}{dt}$) for any time t, then evaluate this rate of change at $t = 1$.

8. White lead (^{210}Pb) is a pigment in oil based paints that can be used to determine the age of a painting and is often used to detect forgeries. The half-life of ^{210}Pb is 22 years. Suppose that a painting is found to have about 5% of its ^{210}Pb remaining. Determine the approximate age of this painting.

9. a. Suppose that a colony of *Escherichia coli* is growing exponentially and satisfies

$$P(t) = 1000e^{0.01t},$$

where t in minutes. Find the doubling time of this population.

b. Suppose a mutant enters the colony ($M(0) = 1$) and satisfies a similar exponential (or Malthusian growth), but it doubles in 25 min. Thus,

$$M(t) = 1e^{kt},$$

where you determine the value of k. Find when the mutant is 20% of the colony.

c. Find both populations at $t = 500$ min and determine the rate of growth (in bacteria/min) of each of these populations at this time.

10. Many biologists in fishery management use Ricker's model to study the population of fish. Let P_n be the population of fish in any year n, then Ricker's model is given by

$$P_{n+1} = R(P_n) = 2.7P_ne^{-0.004P_n},$$

where the parameters are the best fit to a set of data for the number of fish sampled from a particular river.

a. The updating function for this model is

$$R(P) = 2.7Pe^{-0.004P}.$$

Find the derivative of the updating function. Find the P and R-intercepts. Determine any maxima or minima, including both the P and R coordinates. Find any asymptotes, then sketch a graph of this updating function.

b. Equilibria, P_e, satisfy the equation:

$$P_e = R(P_e).$$

Find all equilibria for this Ricker's model.

c. Evaluate the derivative at the equilibria and use the value to determine the stability of all equilibrium points.

11. After the Malthusian growth and logistic growth models, the next most popular discrete population model is the Beverton-Holt model. Let P_n be the population of an animal in any year n, then the Beverton-Holt is given by

$$P_{n+1} = B(P_n) = \frac{4P_n}{1 + 0.002P_n},$$

where the parameters are the best fit to a set of data for some population.

a. The updating function for this model is

$$B(P) = \frac{4P}{1 + 0.002P}.$$

Find the derivative of the updating function. Find the P and B-intercepts. Determine any maxima or minima, including both the P and B coordinates. Find any asymptotes, then sketch a graph of this updating function.

b. Equilibria, P_e, satisfy the equation:

$$P_e = B(P_e).$$

Find all equilibria for this Beverton-Holt model.

c. Evaluate the derivative at the equilibria and use the value to determine the stability of all equilibrium points.

12. Hassell's model is used to study population of insects. Let P_n be the population of a species of beetles in weeks n and suppose that Hassell's model is given by

$$P_{n+1} = H(P_n) = \frac{20P_n}{(1 + 0.004P_n)^4}.$$

where the parameters are the best fit to a set of data for some population.

a. The updating function for this model is

$$H(P) = \frac{20P}{(1 + 0.004P)^4}.$$

Find the derivative of the updating function. Find the P and H-intercepts. Determine any maxima or minima, including both the P and H coordinates. Find any asymptotes, then sketch a graph of this updating function.

b. Equilibria, P_e, satisfy the equation:

$$P_e = H(P_e).$$

Find all equilibria for this Hassell's model.

c. Evaluate the derivative at the equilibria and use the value to determine the stability of all equilibrium points.

13. Consider the chalone model for mitosis given by the equation

$$P_{n+1} = f(P_n) = \frac{2P_n}{1 + (0.05P_n)^2},$$

where P_n is the population density at the n^{th} time step.

a. Find all equilibria of the model, that is, what population densities remain constant for any time period.

b. Differentiate the updating function $f(P_n)$. Find the population density P_n that results in the highest mitotic increase in density at the next time step.

c. Sketch a graph of $f(P)$ for $P \geq 0$, showing the intercepts, all extrema, and any asymptotes.

14. Certain gregarious animals require a minimum number of animals in a colony before they reproduce successfully. This is called the *Allee effect*. Consider the following model for the population of a gregarious bird species, where the population, N_n, is given in thousands of birds:

$$N_{n+1} = N_n + 0.1N_n\left(1 - \frac{1}{9}(N_n - 5)^2\right).$$

a. Assume that the initial population is $N_0 = 4$, then determine the population for the next two generations (N_1 and N_2).

b. Find all equilibria for this model.

c. The model above can be expanded to give

$$N_{n+1} = A(N_n) = \frac{37}{45}N_n + \frac{1}{9}N_n^2 - \frac{1}{90}N_n^3.$$

Find the derivative of $A(N)$. Evaluate the derivative $A'(N)$ at each of the equilibria found above and determine the local behavior of the solution near each of those equilibria.

d. Give a brief biological description of what your results imply about this gregarious species of bird.

15. a. The population of Mexico in 1950 was about 28.49 million, while in 1980, it was about 68.34 million. Assume that the population is growing according to the discrete Malthusian growth equation

$$P_{n+1} = (1 + r)P_n, \quad \text{with} \quad P_0 = 28.49,$$

where P_0 is the population in 1950 and n is in decades. Use the population in 1980 (P_3) to find the value of r (to **4 significant figures**). Find how long it would take for this population to double.

b. Estimate the population in 2000 based on the Malthusian growth model. Given that the population in 2000 was 99.93 million, find the percent error between the actual and predicted values.

c. A better model fitting the census data for Mexico is a logistic growth model given by

$$P_{n+1} = F(P_n) = 1.48P_n - 0.0035P_n^2,$$

where again n is in decades after 1950. Estimate the populations in 1960 and 1970 by computing P_1 and P_2, where $P_0 = 28.49$.

d. Find the equilibrium for this logistic growth model. Calculate the derivative of $F(P)$ and evaluate it at the larger of the equilibria. What does this value say about the behavior of the solution near this equilibrium? Is it stable or unstable? Is it monotonic or oscillatory?

16. The logistic growth model is one of the most common models used in ecological research. Consider a yeast population that satisfies the logistic growth model

$$Y(t) = \frac{1000}{1 + 19e^{-0.1t}},$$

where t is in hours and Y is in yeast/cc.

a. Differentiate this function, then find the second derivative. Use the second derivative to find any points of inflection ($t \geq 0$), giving both the t and Y values.

b. Find any intercepts and asymptotes for $Y(t)$ ($t \geq 0$). Sketch a graph of $Y(t)$. Find how long it takes for the initial population to double.

c. When is $Y(t)$ increasing most rapidly, and what is this rate of increase? Find the intercepts and asymptotes for $Y'(t)$ ($t \geq 0$), then sketch the graph for this function.

d. A Malthusian growth model that approximates this yeast population during the early stages of growth is given by

$$M(t) = 50e^{0.1t}.$$

Find a doubling time from this model, then compare your result to the logistic growth model above.

17. The growth in length of sculpin is approximated by the von Bertalanffy equation

$$L(t) = 16(1 - e^{-0.4t}),$$

where t is in years and L is in cm. An allometric measurement of sculpin shows that their weight can be approximated by the model

$$W(L) = 0.07L^3,$$

where W is in g.

a. Find the intercepts and any asymptotes for the length of a sculpin, then sketch of graph showing the length of a sculpin as it ages.

b. Create a composite function to give the weight of the sculpin as a function of its age, $W(t)$. Find the intercepts and any asymptotes for $W(t)$, then sketch of graph showing the weight of a sculpin as it ages.

c. Find the derivative of $W(t)$ using the chain rule. Also, compute the second derivative, then determine when this second derivative is zero. From this information, find at what age the sculpin are increasing their weight the most and determine what that weight gain is. Be sure to give the units of weight gain.