1. Differentiate the following:

a. $f(x) = 6x^3 + \dfrac{2}{x^2} - e^{2x}(x^2 - 9)$,

b. $g(x) = 2e^{-3x} + \ln(x^2) - 5$,

c. $h(x) = 2x^6 \ln(x) - e^{x^2+4x} + \frac{e^{-4x}}{2}$,

d. $k(t) = \dfrac{1}{4}t^2 - \dfrac{4}{\sqrt{t}} + \dfrac{2 + e^{2t}}{t^2 - 3}$,

e. $p(w) = \dfrac{2\ln(w) + w}{w^3 - 8} - w^{2/5} + \dfrac{1}{w^3}e^{-w}$,

f. $q(z) = Az\ln(z^2) - B\sqrt{z} + \dfrac{C}{z^3}$.

g. $r(x) = e^{2x}(x^3 - 5x + 7)^4 - \dfrac{7x}{\sqrt{x^2 + 2x + 5}}$,

h. $F(y) = (3y^2 - 4y + 6)^5 + \ln(2y + 9)$.

2. For each of the following functions, give the domain. Find all x and y-intercepts and any asymptotes, if they exist. Find the derivative of the functions, then determine any maxima or minima. Give both the x and y values. Write the second derivative and find any points of inflection. Finally, sketch the graph of the function.

a. $y = 27x - x^3$,

b. $y = x^4 - 4x$,

c. $y = x^3 + 3x^2 + 3x + 1$,

d. $y = 18x^2 - x^4$,

e. $y = x + \dfrac{4}{x}$,

f. $y = 4xe^{-0.02x}$,

g. $y = (x + 3)\ln(x + 3)$,

h. $y = (x - 4)e^{2x}$.

i. $y = \dfrac{4x^2}{x + 3}$,

j. $y = \dfrac{2\,e^{2x}}{x - 1}$.

3. Body temperatures of animals undergo circadian rhythms. A subject's temperature is measured from 8 AM until midnight, and his body temperature, T (in °C), is best approximated by the cubic polynomial

$$T(t) = 0.002(t^3 - 45t^2 + 600t + 16000),$$

where t is in hours.

a. Find the rate of change in body temperature $\frac{dT}{dt}$. What is the rate of change in body temperature at noon $t = 12$?

b. Use the derivative to find when the maximum temperature of the subject occurs and when the minimum temperature of the subject occurs. What are the body temperatures at those times? State the intervals of time where the subject's body temperature is decreasing.

4. Over a 7 day period in the summer, data were collected on an algal bloom in the ocean. The population of algae (in thousand/cc), $P(t)$, were best fit by the cubic polynomial

$$P(t) = t^3 - 9t^2 + 15t + 30,$$

where t is in days.

a. Find the rate of change in population per day, $\frac{dP}{dt}$. What is the rate of change in the population on the first day, $t = 2$?

b. Use the derivative to find when the relative minimum and maximum populations of algae occur over the time of the survey. Give the populations at those times. Over what intervals of time is the population increasing?

c. Sketch a graph of this polynomial fit to the population of algae. Show clearly the maximum and minimum populations on your graph and include the populations at the beginning of the survey ($t = 0$) and at the end ($t = 7$).

5. Plants require nutrients to grow, but an excess of any one nutrient can be toxic. Suppose that it is found that the rate of growth of a plant (in mm/day) satisfies the equation:

$$R(n) = \frac{10n}{1 + n^2},$$

where n is the concentration of the essential nutrient (in mg/l).

a. Find the rate of change in growth rate as a function of nutrient application, $\frac{dR}{dn}$. What is the rate of change in the growth rate when $n = 2$?

b. Use the derivative to any relative minima or maxima for $R(n)$ for $n \geq 0$. Give both the n and R values.

c. Find the n and R-intercepts and any asymptotes. Sketch a graph of this function for $n \geq 0$.

6. Some hormones have a strong effect on mood, so finding a delivery device that delivers a hormone at a more constant level over a longer period of time is important for hormone therapy. Suppose that a drug company finds a polymer that can be implanted to deliver a hormone, $h(t)$, which is experimentally found to satisfy

$$h(t) = 40 \left(e^{-0.005t} - e^{-0.15t} \right),$$

where h is in nanograms per deciliter of blood (ng/dl) and t is in days.

a. Find the maximum concentration of this hormone in the body and when this occurs.

b. Determine all intercepts and asymptotes, then graph $h(t)$ for $0 \leq t \leq 150$. Use the graph to approximate how long the hormone level remains above 20 ng/dl.

7. The distribution of seeds from around a plant satisfies an exponential distribution. It is more likely to find seeds close to a plant than it is far away. Suppose that experimental measurements fitting the seed distribution radially from the plant satisfies

$$P(r) = 0.04r\,e^{-0.2r},$$

where P is the probability of finding a seed r meters from the plant. Find the distance r and the probability $P(r)$ at which a seed is most likely to land. That is find the maximum probability from the function above. Sketch a graph of $P(r)$ showing any intercepts, asymptotes, and local extrema.

8. A tumor growing according to Gompertz's model satisfies the growth law

$$G(N) = N(0.8 - 0.04\ln(N)) \text{ (cells/day)},$$

where N is the number of tumor cells and the time units are days.

a. Find the equilibrium number of tumor cells by solving when $G(N) = 0$.

b. Compute $G'(N)$ and determine the population at which the maximum rate of growth of the tumor is occurring. What is the maximum growth rate (in cells/day)?

9. a. After ingestion of a sugar-rich meal, the concentration of glucose rises very rapidly in the blood, then through insulin, the glucose is converted to glycogen for later use as energy. An approximation for the concentration of glucose in the body after this meal is given by

$$g(t) = 70 + 90e^{-0.7t},$$

where t is in hours and g is in mg/100 ml of blood. Find how long it takes for the concentration of glucose in the blood to reach 90 mg/100 ml of blood. Sketch a graph for the concentration of glucose in the blood.

b. Find the rate of change of glucose per hour ($\frac{dg}{dt}$) for any time t, then evaluate this rate of change at $t = 1$.

c. The release of insulin responds to this rise in glucose. Suppose that the level of insulin satisfies the function

$$i(t) = 10(e^{-0.4t} - e^{-0.5t}).$$

Find when the insulin reaches its maximum concentration and determine what its maximum concentration is. Sketch a graph of the insulin concentration.

d. Find the rate of change of insulin per hour ($\frac{di}{dt}$) for any time t, then evaluate this rate of change at $t = 1$.

10. White lead (^{210}Pb) is a pigment in oil based paints that can be used to determine the age of a painting and is often used to detect forgeries. The half-life of ^{210}Pb is 22 years. Suppose that a painting is found to have about 5% of its ^{210}Pb remaining. Determine the approximate age of this painting.

11. a. Suppose that a colony of *Escherichia coli* is growing exponentially and satisfies

$$P(t) = 1000e^{0.01t},$$

where t in minutes. Find the doubling time of this population.

b. Suppose a mutant enters the colony ($M(0) = 1$) and satisfies a similar exponential (or Malthusian growth), but it doubles in 25 min. Thus,

$$M(t) = 1e^{kt},$$

where you determine the value of k. Find when the mutant is 20% of the colony.

c. Find both populations at $t = 500$ min and determine the rate of growth (in bacteria/min) of each of these populations at this time.

12. Many biologists in fishery management use Ricker's model to study the population of fish. Let P_n be the population of fish in any year n, then Ricker's model is given by

$$P_{n+1} = R(P_n) = 2.7P_n e^{-0.004P_n},$$

where the parameters are the best fit to a set of data for the number of fish sampled from a particular river.

a. The updating function for this model is

$$R(P) = 2.7Pe^{-0.004P}.$$

Find the derivative of the updating function. Find the P and R-intercepts. Determine any maxima or minima, including both the P and R coordinates. Find any asymptotes, then sketch a graph of this updating function.

b. Equilibria, P_e, satisfy the equation:

$$P_e = R(P_e).$$

Find all equilibria for this Ricker's model.

c. The stability of an equilibrium is determined by the value of the derivative at that equilibrium. If the value of the derivative is less than 1 (and greater than -1), then the equilibrium is stable. If the derivative is greater than 1 at an equilibrium, it is unstable. Evaluate the derivative at the equilibria and use the value to determine the stability of all equilibrium points.

13. After the Malthusian growth and logistic growth models, the second most popular discrete population model is the Beverton-Holt model. Let P_n be the population of an animal in any year n, then the Beverton-Holt is given by

$$P_{n+1} = B(P_n) = \frac{4P_n}{1 + 0.002P_n},$$

where the parameters are the best fit to a set of data for some population.

a. The updating function for this model is

$$B(P) = \frac{4P}{1 + 0.002P}.$$

Find the derivative of the updating function. Find the P and B-intercepts. Determine any maxima or minima, including both the P and B coordinates. Find any asymptotes, then sketch a graph of this updating function.

b. Equilibria, P_e, satisfy the equation:

$$P_e = B(P_e).$$

Find all equilibria for this Beverton-Holt model.

c. The stability of an equilibrium is determined by the value of the derivative at that equilibrium. If the value of the derivative is less than 1 (and greater than -1), then the equilibrium is stable. If the derivative is greater than 1 at an equilibrium, it is unstable. Evaluate the derivative at the equilibria and use the value to determine the stability of all equilibrium points.

14. Hassell's model is used to study population of insects. Let P_n be the population of a species of beetles in weeks n and suppose that Hassell's model is given by

$$P_{n+1} = H(P_n) = \frac{20P_n}{(1 + 0.004P_n)^4}.$$

where the parameters are the best fit to a set of data for some population.

a. The updating function for this model is

$$H(P) = \frac{20P}{(1 + 0.004P)^4}.$$

Find the derivative of the updating function. Find the P and H-intercepts. Determine any maxima or minima, including both the P and H coordinates. Find any asymptotes, then sketch a graph of this updating function.

b. Equilibria, P_e, satisfy the equation:

$$P_e = H(P_e).$$

Find all equilibria for this Hassell's model.

c. The stability of an equilibrium is determined by the value of the derivative at that equilibrium. If the value of the derivative is less than 1 (and greater than -1), then the equilibrium is stable. If the derivative is greater than 1 at an equilibrium, it is unstable. If the value of the derivative is less than -1, then the equilibrium is unstable and oscillatory. Evaluate the derivative at the equilibria and use the value to determine the stability of all equilibrium points.

15. Consider the chalone model for mitosis given by the equation

$$P_{n+1} = f(P_n) = \frac{2P_n}{1 + (0.05P_n)^2},$$

where P_n is the population density at the n^{th} time step.

a. Find all equilibria of the model, that is, what population densities remain constant for any time period.

b. Differentiate the updating function $f(P_n)$. Find the population density P_n that results in the highest mitotic increase in density at the next time step.

c. Sketch a graph of $f(P)$ for $P \geq 0$, showing the intercepts, all extrema, and any asymptotes.

16. The logistic growth model is one of the most common models used in ecological research. Consider a yeast population that satisfies the logistic growth model

$$Y(t) = \frac{1000}{1 + 19e^{-0.1t}},$$

where t is in hours and Y is in yeast/cc.

a. Differentiate this function, then find the second derivative. Use the second derivative to find any points of inflection ($t \geq 0$), giving both the t and Y values.

b. Find any intercepts and asymptotes for $Y(t)$ ($t \geq 0$). Sketch a graph of $Y(t)$. Find how long it takes for the initial population to double.

c. When is $Y(t)$ increasing most rapidly, and what is this rate of increase? Find the intercepts and asymptotes for $Y'(t)$ ($t \geq 0$), then sketch the graph for this function.

d. A Malthusian growth model that approximates this yeast population during the early stages of growth is given by

$$M(t) = 50e^{0.1t}.$$

Find a doubling time from this model, then compare your result to the logistic growth model above.

17. The growth in length of sculpin is approximated by the von Bertalanffy equation

$$L(t) = 16(1 - e^{-0.4t}),$$

where t is in years and L is in cm. An allometric measurement of sculpin shows that their weight can be approximated by the model

$$W(L) = 0.07L^3,$$

where W is in g.

a. Find the intercepts and any asymptotes for the length of a sculpin, then sketch of graph showing the length of a sculpin as it ages.

b. Create a composite function to give the weight of the sculpin as a function of its age, $W(t)$. Find the intercepts and any asymptotes for $W(t)$, then sketch of graph showing the weight of a sculpin as it ages.

c. Find the derivative of $W(t)$ using the chain rule. Also, compute the second derivative, then determine when this second derivative is zero. From this information, find at what age the sculpin are increasing their weight the most and determine what that weight gain is. Be sure to give the units of weight gain.