1. a. Consider

$$y = x + \frac{4}{x} = x + 4x^{-1}.$$

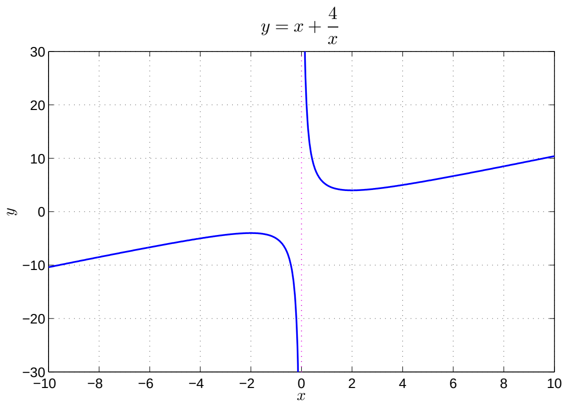
The domain is all $x \neq 0$. This function is not defined at $x = 0$, so $x = 0$ is a vertical asymptote. This function begins with x, so there is no horizontal asymptote (but there is a slant asymptote of $y = x$). Since the function is not defined on the y-axis, there is no y-intercept. If we attempt to solve $y = 0$, then

$$x + \frac{4}{x} = 0 \quad \text{or} \quad x^2 = -4.$$

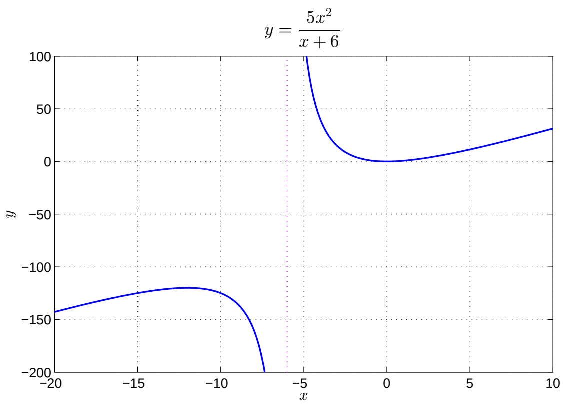
This has no real solutions, so there is no x-intercept. We differentiate this function

$$y'(x) = 1 - 4x^{-2}.$$

Solving $y'(x_c) = 0$ gives $x_c = \pm 2$. The second derivative is $y''(x) = 8x^{-3}$, which is negative for $x < 0$ and positive for $x > 0$. The second derivative test shows that $x_c = -2$ is a relative maximum at $(-2, -4)$. Similarly, the second derivative test shows that $x_c = 2$ is a relative minimum at $(2, 4)$. Below is a graph of this function.



Problem 1a



Problem 1b

b. Consider

$$y = \frac{5x^2}{x + 6}.$$

The domain is all $x \neq -6$, which gives $x = -6$ as a vertical asymptote. Since the power of x in the numerator exceeds the power of x in the denominator, there is no horizontal asymptote. Substituting $x = 0$ into the function gives this function passing through the origin, so there are x and y-intercepts at $x = 0$ and $y = 0$. We differentiate this function using the quotient rule

$$y'(x) = 5\frac{(x + 6)(2x) - x^2(1)}{(x + 6)^2} = 5\frac{x^2 + 12x}{(x + 6)^2}.$$

Solving $y'(x_c) = 0$ gives $x_c^2 + 12x_c = x_c(x_c + 12) = 0$. Thus, the critical points are $x_c = 0$ and $x_c = -12$. It is easy to see that $(0, 0)$ is a relative minimum. Similarly, we see that $(-12, -120)$ is a relative maximum. Above is a graph of this function.

c. Consider

$$y = 4\left(e^{-0.02x} - e^{-0.6x}\right).$$

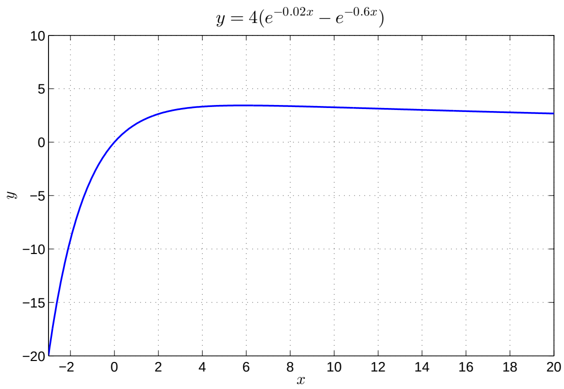
The domain is all x, which says there is no vertical asymptote. Since the exponentials have a negative sign, there is a horizontal asymptote to the right with $y \to 0$ as $x \to +\infty$. Substituting $x = 0$ into the function gives this function passing through the origin, so there are x and y-intercepts at $x = 0$ and $y = 0$. We differentiate this function

$$y'(x) = 4\left(-0.02\, e^{-0.02x} + 0.6\, e^{-0.6x}\right) = 2.4\, e^{-0.6x} - 0.08\, e^{-0.02x}.$$

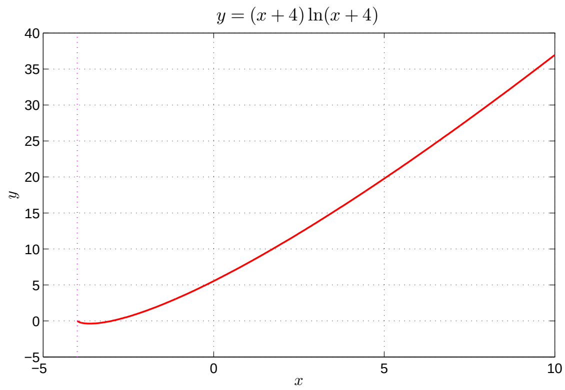
Solving $y'(x_c) = 0$ gives $2.4\, e^{-0.6x_c} = 0.08\, e^{-0.02x_c}$ or

$$e^{0.58x_c} = 30 \quad \text{or} \quad x_c = \frac{\ln(30)}{0.58} \approx 5.86413.$$

This is substituted into the function, giving $y(x_c) \approx 3.43876$. It is easy to see that $(5.86413, 3.43876)$ is a relative maximum (and absolute maximum). Below is a graph of this function.



Problem 1c



Problem 1d

d. Consider

$$y = (x + 4)\ln(x + 4).$$

Since the logarithm function is only defined for its argument greater than zero, the domain is $x > -4$. For this function, the edge of the domain is not a vertical asymptote. The function can be shown numerically (or by other mathematical techniques) that as $x \to -4$, this function tends to zero

$$\lim_{x \to -4} y(x) = 0.$$

This is not an x-intercept. The y-intercept satisfies $y(0) = 4\ln(4) \approx 5.54518$. Solving $(x+4)\ln(x+4) = 0$ implies $\ln(x+4) = 0$ or $x = -3$. Thus, $x = -3$ is the x-intercept. There is no horizontal asymptote. We differentiate this function

$$y'(x) = (x + 4)\left(\frac{1}{x + 4}\right) + \ln(x + 4) = 1 + \ln(x + 4).$$

Solving $y'(x_c) = 0$ gives $1 + \ln(x_c + 4) = 0$ or

$$\ln(x_c + 4) = -1 \quad \text{or} \quad x_c + 4 = e^{-1} \quad \text{or} \quad x_c = e^{-1} - 4 \approx -3.63212.$$

This is substituted into the function, giving $y(x_c) = e^{-1}\ln(e^{-1}) \approx -0.367879$. It is easy to see that $(-3.63212, -0.367879)$ is a relative minimum (and absolute minimum). Above is a graph of this function.

e. Consider

$$y = \frac{2(x-4)}{x^2+9}.$$

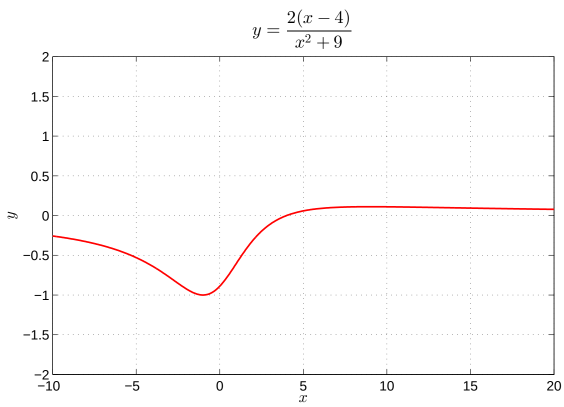
Since the denominator is always positive, the domain is all x, which says there is no vertical asymptote. Since the power of the numerator is less than the power of the denominator, then there is a horizontal asymptote of $y = 0$. Substituting $x = 0$ into the function gives $y(0) = -\frac{8}{9}$, so the y-intercept is $y = -\frac{8}{9}$. The x-intercept satisfies the numerator equal to zero, so $x = 4$. We differentiate this function using the quotient rule

$$y'(x) = 2\frac{(x^2+9)\cdot(1) - (x-4)(2x)}{(x^2+9)^2} = 2\frac{9+8x-x^2}{(x^2+9)^2}.$$

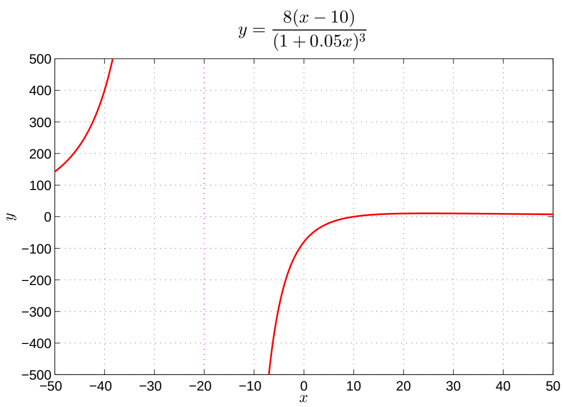
Solving $y'(x_c) = 0$ gives $x_c^2 - 8x_c - 9 = (x_c - 9)(x_c + 1) = 0$ or $x_c = -1$ and $x_c = 9$. Substituting these critical points into the function gives

$$y(-1) = -1 \qquad \text{and} \qquad y(9) = \frac{1}{9}.$$

From these points it is easy to see that $(-1,-1)$ is a relative minimum (and absolute minimum). Also, the point $\left(9, \frac{1}{9}\right)$ is a relative maximum (and absolute maximum). Below is a graph of this function.



Problem 1a



Problem 1b

f. Consider

$$y = \frac{8(x-10)}{(1+0.05x)^3}.$$

Solving the denominator equal to zero gives $x = -20$. It follows that domain is all $x \neq -20$, which says there is a vertical asymptote at $x = -20$. Since the power of the numerator is less than the power of the denominator, then there is a horizontal asymptote of $y = 0$. Substituting $x = 0$ into the function gives $y(0) = -80$, so the y-intercept is $y = -80$. The x-intercept satisfies the numerator equal to zero, so $x = 10$.

We differentiate this function using the quotient rule

$$\begin{aligned} y'(x) &= 8\frac{((1+0.05x)^3\cdot(1)-(x-10)3(1+0.05x)^2(0.05))}{(1+0.05x)^6} \\ &= 8\frac{((1+0.05x)-(x-10)(0.15))}{(1+0.05x)^4} \\ &= 8\frac{(2.5-0.1x)}{(1+0.05x)^4}. \end{aligned}$$

Solving $y'(x_c)=0$ gives $2.5-0.1x_c=0$ or $x_c=25$. Substituting this critical point into the function give

$$y(25)=\frac{8(15)}{2.25^3}\approx 10.53498.$$

From this it is easy to see that $(25, 10.53498)$ is a relative maximum. Above is a graph of this function.

2. a. The differential equation is given by

$$\frac{dw}{dt}=0.02w+4=0.02(w+200).$$

We make the substitution $z(t)=w(t)+200$ or $z(0)=2+200=202$, since $w(0)=2$. The modified differential equation is $z'=0.02z$, which has the solution $z(t)=202e^{0.02t}=w(t)+200$. It follows that

$$w(t)=202e^{0.02t}-200.$$

b. The differential equation is given by

$$\frac{dx}{dt}=3-0.1x=-0.1(x-30).$$

We make the substitution $z(t)=x(t)-30$ or $z(0)=4-30=-26$, since $x(0)=4$. The modified differential equation is $z'=-0.1z$, which has the solution $z(t)=-26e^{-0.1t}=x(t)-30$. It follows that

$$x(t)=30-26e^{-0.1t}.$$

c. This is a linear differential equation, so we first write

$$\frac{dy}{dt}=2+\frac{y}{3}=\frac{1}{3}(y+6).$$

Thus, we make the substitution $z(t)=y(t)+6$, giving the differential equation $\frac{dz}{dt}=\frac{1}{3}z$ with the initial condition $z(0)=y(0)+6=8$. Thus, $z(t)=8\,e^{t/3}$. It follows that

$$y(t)=8\,e^{t/3}-6.$$