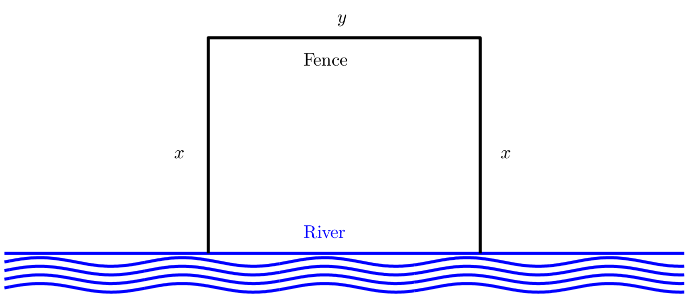
1. a. See the diagram below. Let x be the two sides perpendicular to the river and y be the side parallel to the river. The function to be optimized is the area given by $A(x, y) = xy$. The constraint condition is the perimeter of the area contained by the fence, which satisfies, $P(x, y) = 2x + y = 200$. It follows that we can write $y = 200 - 2x$. Thus, the function we need to optimize is

$$A(x) = x(200 - 2x) = 200x - 2x^2.$$

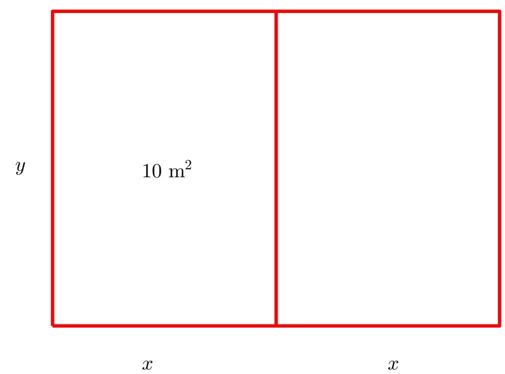
Differentiating:

$$A'(x) = 200 - 4x.$$

This yields a critical point (maximum) at $x_c = 50$. It follows that the optimal dimensions for maximizing the area of this region are $x = 50$ m and $y = 100$ m, yielding an maximal area of $A(50) = 5000$ m^2.



Problem 1a



Problem 1b

b. See the diagram above. Let x be the side of one study plot area and y be the other side of the study plot area, where y is the dimension of the shared side. We are optimizing the perimeter, which satisfies the function, $P(x, y) = 4x + 3y$. The constraint condition in this problem is that the area of each plot is 10 m^2, so $A(x, y) = xy = 10$. This can be solved to give $y = \frac{10}{x}$. We substitute this into the perimeter equation to give:

$$P(x) = 4x + \frac{30}{x} = 4x + 30x^{-1}.$$

Upon differentiation, we see

$$P'(x) = 4 - 30x^{-2}.$$

for the critical point, this is set equal to zero, so

$$4 - \frac{30}{x^2} = 0 \quad \text{or} \quad x = \sqrt{7.5} \approx 2.7386 \text{ m} \quad \text{and} \quad y = \frac{10}{\sqrt{7.5}} \approx 3.6515 \text{ m}.$$

The minimum amount of fence needed is

$$P(2.7386) = 4(2.7386) + \frac{30}{2.7386} \approx 21.9089 \text{ m}.$$

c. See the diagram below. Let the width of the box be x, the length be $4x$, and the height be y. The function to be optimized for maximum volume is $V(x,y) = 4x^2y$. Since this is an open box, the constraint condition is the surface area, $S(x,y) = 4x^2 + 10xy = 1200$ cm^2. This equation is solved for y giving:

$$y = \frac{1200 - 4x^2}{10x} = \frac{120}{x} - \frac{2x}{5}.$$

The volume is written as a function of only the variable x,

$$V(x) = 4x^2\left(\frac{120}{x} - \frac{2x}{5}\right) = 480x - \frac{8}{5}x^3.$$

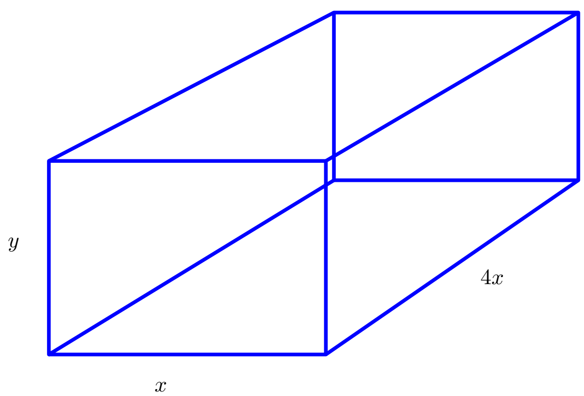
This function is differentiated to give

$$V'(x) = 480 - \frac{24}{5}x^2.$$

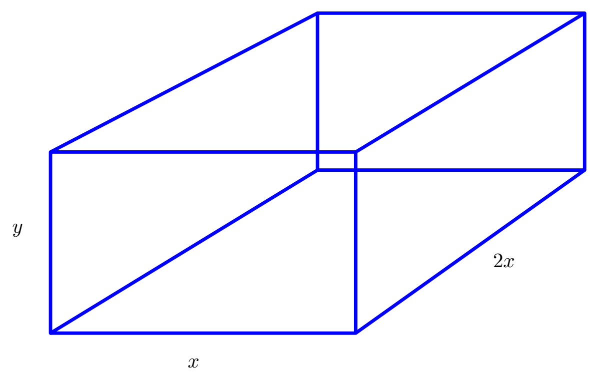
To find the critical x, we solve

$$480 - \frac{24}{5}x_c^2 = 0 \quad \text{or} \quad x_c^2 = 100 \quad \text{or} \quad x_c = 10\,\text{cm}.$$

Substituting back, we find $y_c = 8$ cm and the maximum volume is $V(10) = 3200$ cm^3.



Problem 1c



Problem 1d

d. See diagram above. The objective function for this problem is the minimization of the surface area, $S(x,y) = 4x^2 + 6xy$. The constraint is that the box must have a volume satisfying, $V(x,y) = 2x^2y = 5000$ cm^3. This constraint can be solved for y to give $y = \frac{5000}{2x^2} = \frac{2500}{x^2}$. We substitute y into S to obtain:

$$S(x) = 4x^2 + \frac{15,000}{x} = 4x^2 + 15,000x^{-1}.$$

This is differentiated to yield:

$$S'(x) = 8x - 15000x^{-2} = 8x - \frac{15,000}{x^2}.$$

Setting the derivative equal to zero for the critical point, we find:

$$8x_c - \frac{15,000}{x_c^2} = 0 \quad \text{or} \quad x_c^3 = \frac{15,000}{8} = 1875.$$

It follows that $x_c = 1875^{1/3} \approx 12.3311$ cm and $y = \frac{2500}{1875^{2/3}} \approx 16.4414$ cm. From the formula for the surface area, we obtain the minimum surface area is

$$S(x_c) = 4(1875^{2/3}) + \frac{15,000}{1875^{1/3}} \approx 1824.7\,\text{cm}^2.$$

e. See diagram below. The objective function for this problem is the minimization of the surface area, $S(r, y) = \pi r^2 + 2\pi r y$. The constraint is that the box must have a volume satisfying, $V(r, y) = \pi r^2 y = 4500$ cm^3. This constraint can be solved for y to give $y = \frac{4500}{\pi r^2}$. We substitute y into S to obtain:

$$S(r) = \pi r^2 + \frac{9000}{r} = \pi r^2 + 9000r^{-1}.$$

This is differentiated to yield:

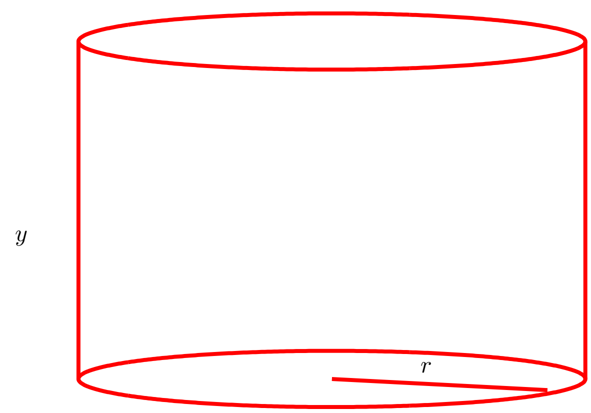
$$S'(r) = 2\pi r - 9000r^{-2} = 2\pi r - \frac{9000}{r^2}.$$

Setting the derivative equal to zero for the critical point, we find:

$$2\pi r_c - \frac{9000}{r_c^2} = 0 \quad \text{or} \quad r_c^3 = \frac{4500}{\pi}.$$

It follows that $r_c = \left(\frac{4500}{\pi}\right)^{1/3} \approx 11.2725$ cm and $y \approx \frac{4500}{\pi(11.2725)^2} \approx 11.2725$ cm. From the formula for the surface area, we obtain the minimum surface area is

$$S(r_c) \approx \pi(11.2725)^2 + \frac{9000}{11.2725} \approx 1197.6\,\text{cm}^2.$$



2. a. If $\sin(\theta) = \frac{3}{4}$, then we can assume the opposite side is 3, while the hypotenuse is 4. By Pythagorean's Theorem the adjacent side is $x = \sqrt{4^2 - 3^2} = \sqrt{7}$. It follows that

$$\cos(\theta) = \frac{\sqrt{7}}{4}.$$

Using a scientific calculator, we find $\theta = \arcsin\left(\frac{3}{4}\right) = 0.8480621$ radians.

b.If $\cos(\theta) = \frac{2}{7}$, then we can assume the adjacent side is 2, while the hypotenuse is 7. By Pythagorean's Theorem the adjacent side is $x = \sqrt{7^2 - 2^2} = \sqrt{45} = 3\sqrt{5}$. It follows that

$$\sin(\theta) = \frac{3\sqrt{5}}{7}.$$

Using a scientific calculator, we find $\theta = \arccos\left(\frac{2}{7}\right) = 1.281045$ radians.

3. a. The damped spring-mass system satisfying

$$y(t) = 12 + 7\,e^{-0.3t}\sin(5t),$$

passes through $y(t) = 12$ whenever $\sin(5t) = 0$. This occurs whenever $5t = n\pi$ for any $n = 0, 1, \ldots.$ Thus, $y(t_n) = 12$ for $t_n = \frac{n\pi}{5}$ with $n = 0, 1, \ldots.$

b. The velocity of the mass satisfies, $v(t) = y\ (t)$, so

$$v(t) = y\,'(t) = 7(e^{-0.3t}(5\cos(5t)) - 0.3\,e^{-0.3t}\sin(5t)) = 7\,e^{-0.3t}(5\,\cos(5t)) - 0.3\,\sin(5t)).$$

c. The absolute maximum occurs for the first t_m, where $v(t_m) = 0$. From the formula above, this occurs when

$$5\,\cos(5t_m)) - 0.3\,\sin(5t_m) = 0 \qquad \text{or} \qquad \tan(5t_m) = \frac{50}{3}.$$

It follows that

$$5t_m = \arctan\left(\frac{50}{3}\right) \qquad \text{or} \qquad t_m \approx 0.302174.$$

Thus, the absolute maximum is

$$y(t_m) = 12 + 7\,e^{-0.3t_m}\sin(5t_m) \approx 18.38187.$$

The absolute minimum occurs at $t_n = t_m + \frac{\pi}{5} \approx 0.930492$, so

$$y(t_n) = 12 + 7\,e^{-0.3t_n}\sin(5t_n) \approx 6.71451.$$

A graph for the position of this mass for $t \in [0, \pi]$ is shown below.