December 16, 2009 Math 122b Name Key

1. (26pts) Evaluate the following integrals:

a. $\int (4 - e^{-0.2t})(1 - 2e^{-0.4t})\,dt$,

$= \int (4 - 8e^{-0.4t} - e^{-0.2t} + 2e^{-0.6t})\,dt$

b. $\int \frac{6\sin(\ln(x)+2)}{x}\,dx$, $= 6\int \sin(u)\,du$

$u = \ln(x) + 2$

$du = \frac{dx}{x}$

$= -6\cos(u) + C$

6, 6 a. $4t + 20e^{-0.4t} + 5e^{-0.2t} - \frac{10}{3}e^{-0.6t} + C$ b. $-6\cos(\ln(x)+2) + C$

c. $\int_0^{\pi/3} (9\sin(3t) + 4)\,dt$,

$= -3\cos(3t) + 4t \Big|_0^{\pi/3}$

$= 3 + \frac{4\pi}{3} - (-3)$

d. $\int_2^9 \frac{6}{\sqrt{3x-2}}\,dx$, $= 2\int_4^{25} u^{-1/2}\,du$

$u = 3x - 2$

$du = 3dx$

$= 4u^{1/2}\Big|_4^{25} = 4(5-2)$

7, 7 c. $6 + \frac{4\pi}{3} = 10.188$ d. 12

2. (14pts) Solve the following initial value problems:

a. $\frac{dy}{dt} = (0.6t + 7)y$, $y(0) = 25$,

$\int \frac{dy}{y} = \int (0.6t + 7)\,dt$

$\ln(y) = 0.3t^2 + 7t + C$

$y(t) = e^C e^{0.3t^2 + 7t}$

$y(0) = e^C = 25$

8 $y(t) = 25e^{0.3t^2 + 7t}$

b. $\frac{dy}{dt} = 12 + 0.4y$, $y(0) = 20$,

$\frac{dy}{dt} = 0.4(y + 30)$

$z = y + 30$

$\frac{dz}{dt} = \frac{dy}{dt}$

$\frac{dz}{dt} = 0.4z$, $z(0) = 50$

$z(t) = 50e^{0.4t} = y(t) + 30$

6 $y(t) = 50e^{0.4t} - 30$

3. (21pts) Consider the curves

$$y = 4x - 16 \qquad \text{and} \qquad y = 4x - x^2.$$

a. Find all the $x$ and $y$-intercepts for both curves. Determine the slope of the line and the vertex of the parabola. Find the points of intersection, then sketch the graph of these curves. (Label the points clearly.)

1, 1, 1 **Line:** $x$-intercept $\underline{4}$ $y$-intercept $\underline{-16}$ Slope $m = \underline{4}$

2, 1, 2 **Parabola:** $x$-intercepts $\underline{0,\ 4}$ $y$-intercept $\underline{0}$ Vertex $\underline{(2, 4)}$

2, 2 **Points of Intersection:** $\underline{(-4, -32)}$ and $\underline{(4, 0)}$

**GRAPH:**

$4x - 16 = 4x - x^2$

$x^2 - 16 = 0$

$x = \pm 4$

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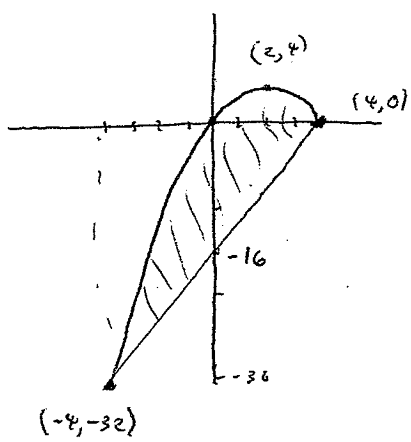



b. Set up and solve the integral that determines the closed area below the parabola and above the line.

7

$$\int_{-4}^{4} \left[(4x - x^2) - (4x - 16)\right] dx = \int_{-4}^{4} (16 - x^2)\, dx$$

$$= 16x - \frac{x^3}{3}\Big|_{-4}^{4} = \left(64 - \frac{64}{3}\right) - \left(-64 + \frac{64}{3}\right)$$

Area = $\underline{\frac{256}{3}}$ $\approx 85.333$

4. (18pts) a. Biological technicians have to worry about the exposure to radioactive sources used in molecular experiments. Radioactive phosphorus, ${}^{32}P$, is often used in experiments with RNA or DNA and has a half-life of 14 days. A differential equation describing the radioactive decay of ${}^{32}P$ is given by:

$$\frac{dR}{dt} = -kR, \qquad R(0) = 50.$$

$R(t) = 50e^{-kt}$

Find $k$ and write the solution to this differential equation.

$e^{14k} = 2$

$k = \frac{\ln(2)}{14}$

3,2 $k =$ 0.04951 $R(t) =$ $50\,e^{-0.04951t}$

b. Suppose a technician is receiving an exposure from mislaid sample of ${}^{32}P$ with

$$D(t) = 7.5e^{-kt},$$

in mCi/day. The total exposure over 15 days is given by the integral

$$\int_0^{15} D(t)\,dt.$$

Find this total exposure.

$7.5\int_0^{15} e^{-0.04951t}\,dt = -\frac{7.5}{0.04951}\left(e^{-0.04951t}\Big|_0^{15}\right.$

$= -151.483\left(e^{-0.04951(15)} - 1\right) = 151.483\left(1 - e^{-0.74266}\right)$

6 Total Exposure = 79.40

c. How long can the technician stay near this source if the exposure is to be kept to less than 25 mCi?

$25 = \int_0^t 7.5e^{-0.04951t}\,dt = -151.483\left(e^{-0.04951t} - 1\right)$

$151.483e^{-0.04951t} = 126.483$

$t = \frac{1}{0.04951}\ln\left(\frac{151.483}{126.483}\right)$

7

Exposure Time = 3.643 days

5. (25pts) In lab we investigated the length of daylight as the year progresses. We let January 1 be $t = 0$ and took the typical year to be 365.25 days. If we consider New Orleans, Louisiana, the Naval Observatory computes the longest day to have 14.1 hours of daylight. This chart shows that the least amount of daylight is 10.2 hours, and this occurs at $t = 354$ (December 21).

a. Consider a model of daylight hours for New Orleans given by

$$L(t) = A + B\cos(\omega(t - \phi)),$$

$A \approx \dfrac{14.1 + 10.2}{2}$

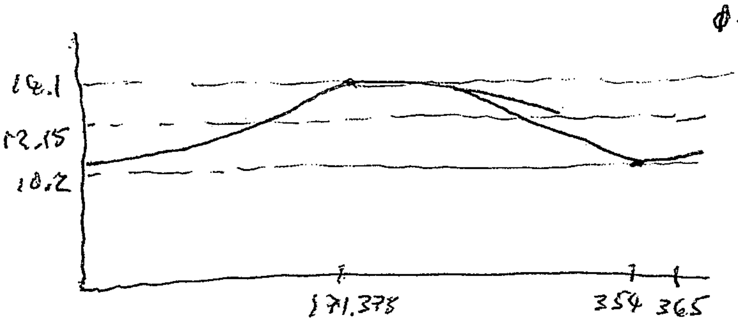
$2\pi \approx \omega \cdot 365.25$

where $A$, $B$, $\omega$, and $\phi$ are constants and $t$ is in days. Use the data above to find the four parameters, then sketch a graph for the daylight hours for New Orleans over one year.

$\omega = \dfrac{2\pi}{365.25}$

**GRAPH:** $\omega(354 - \phi) = \pi$

$\phi = 354 - \dfrac{365.25}{2}$




3 $A =$ 12.15

3 $B =$ 1.95

3 $\omega =$ 0.017202

3 $\phi =$ 171.375

1 $L(t) = 12.15 + 1.95\cos(0.017202(t - 171.375))$

b. Find the number of hours of daylight for September 1 ($t = 243$) according to the model? Find the derivative of $L(t)$ and give its value for September 1 ($t = 243$).

1 $L(243) =$ 12.798 hr

3 $L'(t) = -0.033544\sin(0.017202(t - 171.375))$

1 $L'(243) =$ $-0.03164$ hr/day

c. Find the average length of day for the first 100 days of the year. The average length of day is computed by the following integral:

$$\frac{1}{100}\int_0^{100} L(t)\,dt.$$

$$\frac{1}{100}\left[12.15t + \frac{1.95}{0.017202}\sin(0.017202(t - 171.375))\right]\Bigg|_0^{100}$$

7

$$= 12.15 - \frac{1.95}{1.7202}\left(\sin(0.017202(-71.375)) - \sin(0.017202(-171.375))\right)$$

Average = 11.301

6. (28pts) a. A study on the American Cockroach, *Periplaneta americana*, with a fix amount of food gives a best fitting logistic growth model of the form

$$P_{n+1} = F(P_n) = 2.68\,P_n - 0.0042\,P_n^2, \qquad P_0 = 8,$$

where $n$ is in weeks. Use this model to find the estimated populations in the next two weeks ($P_1$ and $P_2$).

1, 1 $P_1 =$ 21.171 $P_2 =$ 54.856

b. Find all equilibria and compute the derivative of $F(P)$. Determine the stability of the equilibria. Justify your stability argument by evaluating the derivative of the updating function.

$P_e = 2.68 P_e - 0.0042 P_e^2$ $\qquad P_e = 0$ or $P_e = \frac{1.68}{0.0042} = 400$

2 $F'(P) =$ $2.68 - 0.0084 P$

1, 1 $P_{1e} =$ 0 $F'(P_{1e}) =$ 2.68

1, 1 Stable or **Unstable** **Monotonic** or Oscillatory

2, 2 $P_{2e} =$ 400 $F'(P_{2e}) =$ $-0.68$

1, 1 **Stable** or Unstable Monotonic or **Oscillatory**

c. An alternate model that is popular in ecology is Ricker's model. The best fitting version of this model for the data is given by:

$$P_{n+1} = R(P_n) = 2.88\,P_n e^{-P_n/380}, \qquad P_0 = 8.$$

Find all equilibria and compute the derivative of $R(P)$. Determine the stability of the equilibria. Justify your stability argument by evaluating the derivative of the updating function.

$P_e = 2.88 P_e e^{-P_e/380} \Rightarrow P_e = 0$ or $e^{P_e/380} = 2.88$

$P_e = 380 \ln(2.88)$

3 $R'(P) =$ $2.88 e^{-P/380}(1 - P/380)$

1, 1 $P_{1e} =$ 0 $R'(P_{1e}) =$ 2.88

1, 1 Stable or **Unstable** **Monotonic** or Oscillatory

3, 2 $P_{2e} =$ 401.96 $R'(P_{2e}) =$ $-0.05779$

1, 1 **Stable** or Unstable Monotonic or **Oscillatory**

7. (28pts) a. A cat is sitting on a ledge that is 300 cm above the ground. The cat jumps with just the right velocity to catch a pigeon flying 90 cm above its head. If the acceleration due to gravity is 980 cm/sec$^2$, then the initial velocity, $v_0$, of the cat satisfies the differential equation:

$$\frac{dv}{dt} = -980, \qquad v(0) = v_0,$$

with the height of the cat, $h(t)$, satisfying $\frac{dh}{dt} = v(t)$. Find $v(t)$ and $h(t)$, then determine the cat's initial upward velocity, $v_0$. Also, determine the time the cat is in the air and what is the cat's velocity on impact with the ground.

$v(t) = -980t + v_0$ $\qquad h(t) = -490t^2 + v_0 t + 300$

$t_{max} \Rightarrow v(t_{max}) = 0 \Rightarrow t_{max} = \frac{v_0}{980}$

$h(t_{max}) = 390 = 300 + v_0\left(\frac{v_0}{980}\right) - 490\left(\frac{v_0}{980}\right)^2 \Rightarrow \frac{v_0^2}{1960} = 90$

$v_0 = \sqrt{90 \cdot 1960} = 420$ $\qquad h(t) = 0 \quad t = \frac{420 + \sqrt{(420)^2 + 4(300)(490)}}{2(490)}$

5 $\quad v_0 =$ 420 cm/sec

2, 2 $\quad v(t) =$ $420 - 980t$ $\qquad h(t) =$ $300 + 420t - 490t^2$

2, 2 $\quad$ Time in Air = 1.3207 sec $\qquad$ Velocity of impact = $-874.3$ cm/sec

b. You discover the dead pigeon at 10 AM and find its body temperature is 37.0°C. Forty minutes later the body temperature is 35.2°C. If a typical pigeon has a body temperature of 42.0°C and the temperature outside is 16°C, then set up a linear differential equation (with Newton cooling constant $k$) to determine the temperature of the pigeon at any time after its demise. Solve this differential equation, including finding the cooling constant $k$. (Use $t$ in minutes to determine $k$.) Determine the time of death of the pigeon.

$\begin{array}{r} 37 \\ 16 \\ \hline 21 \end{array}$

$z = H - 16 \qquad \frac{dz}{dt} = -kz, \quad z(0) = 21$

$H(t) = 16 + 21e^{-kt}, \qquad H(40) = 16 + 21e^{-40k} = 35.2$

$e^{40k} = \frac{21}{19.2} \qquad k = \frac{1}{40}\ln\left(\frac{21}{19.2}\right)$

3 $\quad \frac{dH}{dt} =$ $-k(H - 16)$

5, 4 $\quad H(t) =$ $16 + 21e^{-0.0022403t}$ $\qquad k =$ 0.0022403

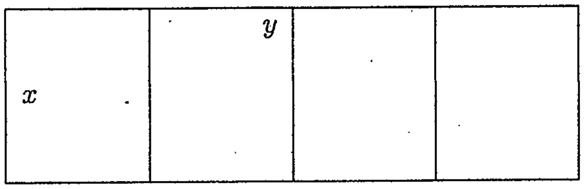
3 $\quad$ Time of Death = $-95.33$ min

8:24.6 AM

$42 = 16 + 21e^{-0.0022403t}$

$t = \frac{1}{k}\ln\left(\frac{21}{26}\right) = -95.33$

8. (16pts) You are an impoverished graduate student in ecology and must design an experimental plot for study with 4 identical 1 $m^2$ sections (so the total area is 4 $m^2$) that are fenced in according to the design below. Define $x$ to be the width of the plot and $y$ be the length of the plot (total length of fencing on the top and bottom of the diagram below). Find the dimensions of the plot that minimizes the amount of fencing that you need to buy for your study area. How much fencing should you buy (in meters)?




3 **Objective Function** $P = 2y + 5x$

3 **Constraint Condition** $A = xy = 4$ $\qquad y = 4/x$

$$P(x) = \frac{8}{x} + 5x$$

$$P'(x) = -\frac{8}{x^2} + 5 \overset{opt.}{\Longrightarrow} 5x^2 - 8 = 0$$

$$5x^2 = 8$$

$$x = \sqrt{8/5}$$

5, 3 $x =$ 1.2649 m and $y =$ 3.1623 m

2 Minimum total length of fencing = 12.649 m

9. (24pts) a. An initially clean pond ($c(0) = 0$) maintains a constant volume of $V = 4000$ m$^3$ of water. There is a stream entering this pond with fertilizer from the surrounding fields. The stream has a flow rate of $f_1 = 60$ m$^3$/day with a fertilizer concentration of $Q_1 = 8\,\mu$g/m$^3$. Assume that this is a well-mixed pond with a stream flowing out at the same rate as the stream flowing in. Write a differential equation describing the concentration of fertilizer in the pond ($c(t)$) and solve this differential equation. Sketch a graph of the solution and determine when the fertilizer concentration reaches 2 $\mu$g/m$^3$.

**GRAPH:**

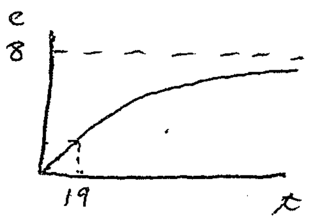
$\frac{1}{4000}\left(\frac{da}{dt} = 60\cdot 8 - 60c\right)$

$\frac{dc}{dt} = -0.015(c-8)$

$z(t) = c(t) - 8$

$\frac{dz}{dt} = -0.015z$, $\quad z(0) = -8$

$z(t) = -8e^{-0.015t}$




2

2 $\quad \frac{dc}{dt} =$ $0.015(8-c)$

$2 = 8 - 8e^{-0.015t}$

5 $\quad c(t) =$ $8 - 8e^{-0.015t}$

$e^{0.015t} = 8/6$

2 $\quad c(t) = 2$ when $t =$ 19.179 days

$t = \frac{1}{0.015}\ln(8/6)$

b. The influx of nutrient to the pond results in the growth of an algae. The concentration of algae, $A(t)$, in the pond satisfies the differential equation:

$$\frac{dA}{dt} = kc(t)(A(t))^{3/4}, \qquad A(0) = 0,$$

where $k = 0.02$ and $c(t)$ is the concentration of fertilizer found in Part a. Solve this differential equation and evaluate $A(100)$.

$$\int A^{-3/4}\,dA = \int\left(0.16 - 0.16e^{-0.015t}\right)dt$$

$$4A^{1/4} = 0.16t + \frac{32}{3}e^{-0.015t} + 4C$$

$$A(t) = \left(0.04t + \frac{8}{3}e^{-0.015t} + C\right)^4$$

$$A(0) = 0 = \left(\frac{8}{3} + C\right)^4 \Rightarrow C = -8/3$$

11 $\quad A(t) =$ $\left(0.04t + \frac{8}{3}e^{-0.015t} - \frac{8}{3}\right)^4$

2 $\quad A(100) =$ 13.827