Spring 2011 Math 121a Name Key

1. (22pts) Differentiate the following functions (you **don't** have to simplify):

a. $f(t) = 7t\,e^{-t^3} - \frac{4}{e^{2t}} + 5(t^4 + \ln(t))^4.$ $= 7te^{-t^3} - 4e^{-2t} + 5(t^4 + \ln(t))^4$

5, 2, 4 $f'(t) = 7\left(te^{-t^3}(-3t^2) + e^{-t^3}\right) + 8e^{-2t} + 20(t^4 + \ln(t))^3(4t^3 + 1/t)$

b. $g(x) = \frac{x^2+5}{e^{2x}+7x} + 4\sqrt{x} - 3\ln(4+x^4).$

5, 2, 4 $g'(x) = \frac{(e^{2x}+7x)(2x) - (x^2+5)(2e^{2x}+7)}{(e^{2x}+7x)^2} + 2x^{-1/2} - 3\frac{4x^3}{4+x^4}$

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2. (11pts) **a.** The population of the India in 1980 was about 692 million, and a census in 1990 showed that the population had grown to 853 million. Assume that this population grows according to the Malthusian growth law,

$$I_{n+1} = (1+r)I_n,$$

where n is the number of decades after 1980, and I_n is population n decades after 1980. Use the data above to find the growth constant r. How long does it take for India's population to double?

$853 = (1+r)692$ $1 + r = \frac{853}{692} = 1.23266$ $n_d = \frac{\ln(2)}{\ln(1+r)} = 3.314$

3, 2 $r =$ 0.23266 Doubling time = 33.14 years

b. In 1980, the population of China was 985 million, while in 1990, it had grown to 1,137 million. Assume China's population is also growing according to a Malthusian growth law.

$$C_{n+1} = (1+s)C_n.$$

Find its rate of growth per decade, s, and predict the year when India's population is equal to China's?

$1+s = \frac{1137}{985} = 1.1543$

$692(1.23266)^n = 985(1.1543)^n$

$\left(\frac{1.23266}{1.1543}\right)^n = \frac{985}{692}$ $n = \frac{\ln(985/692)}{\ln(1.23266/1.1543)}$

$= 5.376$ decades

3, 3 $s =$ 0.1543 Year when populations same = 2033.76

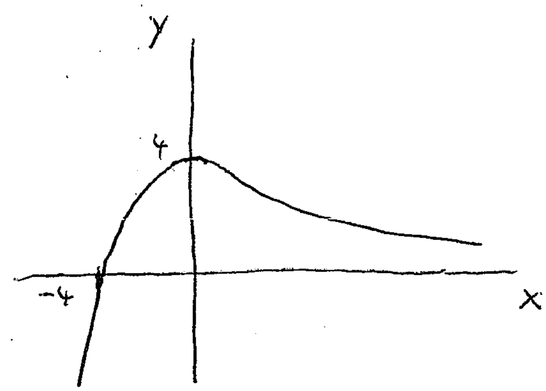
$$y' = (x+4)e^{-x/4}(-\tfrac{1}{4}) + e^{-x/4} \cdot 1 = e^{-x/4}(-\tfrac{x}{4})$$

3. (32pts) Sketch the graph of the following functions. Give the x and y-intercepts, and any asymptotes. Find the derivative and its critical point(s) (including the x and y values). Indicate whether it is a local maxima or minima. For function. (If the function does not have a particular asymptote, extrema or x or y-intercept, indicate "NONE").

a. $y = (x+4)e^{-x/4}$.

Graph of $y(x)$:

2



1 x-intercept(s) -4

1 y-intercept 4

1 Vertical asymptote(s) None

2 Horizontal asymptote(s) 0 as $x \to +\infty$

4 $y'(x) = -\frac{x}{4}e^{-x/4}$

2, 2 $x_{max} =$ 0 $y(x_{max}) =$ 4

½, ½ $x_{min} =$ None $y(x_{min}) =$ None

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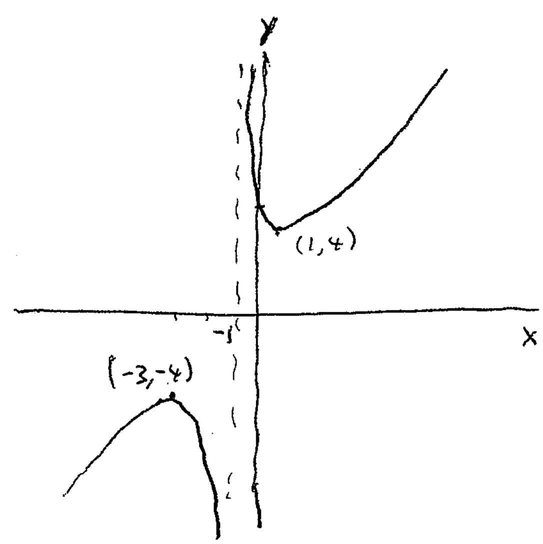
b. $y = \dfrac{x^2+2x+5}{x+1}$.

Graph of $y(x)$:

2



1 x-intercept(s) None

1 y-intercept is 5

2 Vertical Asymptote(s) $x = -1$

1 Horizontal Asymptote(s) None

4 $y'(x) = \dfrac{x^2+2x-3}{(x+1)^2} = \dfrac{(x+3)(x-1)}{(x+1)^2}$

2, ½ $x_{max} =$ -3 $y(x_{max}) =$ -4

2, ½ $x_{min} =$ 1 $y(x_{min}) =$ 4

$$y' = \frac{(x+1)(2x+2) - (x^2+2x+5)}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2}$$

4. (25pts) A study of birds in flight [1] showed an allometric relationship between the weight of the bird and the speed of its flight. The Common Swift (*Apus apus*) had a weight (M) of 0.038 kg and flew at a velocity (U) of 10.6 m/sec. The Common Eider (*Somateria mellissima*) had a weight (M) of 2.015 kg and flew at a velocity (U) of 17.9 m/sec. (Give all answers to **4 significant figures.**)

a. A linear model is given by $U = kM + b$ for some constants k and b. Find the constants k and b.

$$k = \frac{17.9 - 10.6}{2.015 - 0.038} = 3.6925 \qquad b = 17.9 - k(2.015) = 10.460$$

3, 3 $k =$ 3.6925 $b =$ 10.46

b. An allometric model satisfies the relationship given by $U = cM^a$ for some constants c and a. Find the constants c and a.

$$\ln(U) = \ln(c) + a\ln(M)$$

M	ln(M)	U	ln(U)
0.038	-3.2702	10.6	2.36085
2.015	0.70062	17.9	2.8848

$$a = \frac{\ln(17.9) - \ln(10.6)}{\ln(2.015) - \ln(0.038)}$$

$$\ln(c) = \ln(17.9) - a\ln(2.015) = 2.792$$

$$c = 16.319$$

5, 4 $c =$ 16.319 $a =$ 0.13195

c. The Mistle Thrush (*Turdus viscivorus*) weighs 0.114 kg, so use each of the above models to predict the velocity, U, at which the Mistle Thrush flies. If it is clocked at a speed of 11.9 m/sec, then determine the percent error from each of the models (assuming that the actual clocked value is the best).

$$U = 3.6925\,M + 10.46 \qquad U = 16.319\,M^{0.13195}$$

$$100\,\frac{(10.88 - 11.9)}{11.9} = -8.56 \qquad 100\,\frac{(12.25 - 11.9)}{11.9} = 2.97$$

1, 1 Linear Model $U =$ 10.88 m/sec Percent Error = - 8.56%

1, 1 Allometric Model $U =$ 12.25 m/sec Percent Error = 2.97%

d. The Rook (*Corvus frugilegus*) was clocked with a speed of 13.0 m/sec, so use each of the above models to predict the mass, M, of the Rook. If its weight is measured at 0.488 kg, then determine the percent error from each of the models (assuming that the measured weight is the best).

$$M = \frac{U - 10.46}{3.6925} \qquad M = \left(\frac{U}{16.319}\right)^{1/0.13195}$$

$$100\,\frac{(0.6879 - 0.488)}{0.488} = 40.96 \qquad 100\,\frac{(0.1785 - 0.488)}{0.488}$$

2, 1 Linear Model $M =$ 0.6879 kg Percent Error = 40.96%

2, 1 Allometric Model $M =$ 0.1785 kg Percent Error = -63.42%

[1] T. Alerstam, M. Rosén, J. Bäckman, PGP Ericson, O. Hellgren (2007) Flight speeds among bird species: Allometric and phylogenetic effects. *PLoS Biol* 5(8): e197. doi:10.1371

5. (20pts) **a.** An invasive species of insect enters an ecological study area. The initial survey of the region finds a population density of 15.6 insects/m^2. Three weeks later a survey finds a density of 35.7 insects/m^2. Assume that the population is growing according to the discrete Malthusian growth equation

$$P_{n+1} = (1+r)P_n, \quad \text{with} \quad P_0 = 15.6,$$

where P_0 is the initial population and n is in weeks. Use the second survey ($P_3 = 35.7$) to find the value of r (to 4 **significant figures**). Find how long it would take for this population to double.

$P_n = 15.6(1+r)^n$ $\quad 35.7 = 15.6(1+r)^3$ $\quad 1+r = \left(\frac{35.7}{15.6}\right)^{1/3}$

$n_d = \frac{\ln(2)}{\ln(1+r)}$

2, 2 $\quad r =$ 0.3178 $\quad$ Doubling time = 2.512 (in weeks)

b. Estimate the population after 4 weeks based on the Malthusian growth model. Another survey finds that the population after weeks was 44.6 insects/m^2, find the percent error between the actual and predicted values.

$P_4 = 15.6(1.3178)^4$ $\quad 100\,\frac{(P_4 - 44.6)}{44.6}$

2, 2 $\quad P_4 =$ 47.05 $\quad$ and $\quad$ Percent Error = 5.48%

c. A better model fitting the survey data is a logistic growth model given by

$$P_{n+1} = F(P_n) = 1.42P_n - 0.00488P_n^2,$$

where again n is in weeks. Estimate the populations P_1 and P_2 given that $P_0 = 15.6$.

1, 1 $\quad P_1 =$ 20.96 $\quad$ and $\quad P_2 =$ 27.62

d. Find the equilibria for this logistic growth model. Calculate the derivative of $F(P)$ and evaluate it at the larger of the equilibria. What does this value say about the behavior of the solution near this equilibrium?

$P_e = 1.42P_e - 0.00488P_e^2$ $\quad P_e = 0$ or $P_e = \frac{0.42}{0.00488}$

1, 2 $\quad P_{1e} =$ 0 $\quad$ and $\quad P_{2e} =$ 86.066 $\quad (P_{1e} < P_{2e})$

3, 2 $\quad F'(P) =$ $1.42 - 0.00976P$ $\quad F'(P_{2e}) =$ 0.58

1, 1 $\quad$ **Stable** or **Unstable** $\quad$ **Monotonic** or **Oscillatory**

6. (20pts) **a.** A man with a chronic lung problem breathes a supply of air enriched with helium (550 ppm). The initial concentration of helium in his lungs is $c_0 = 550$, and the measurement of helium in his lungs after his first breath is $c_1 = 500$. If the concentration of helium in the room is negligible, then an appropriate model for the concentration of helium (He) is given by the model:

$$c_{n+1} = (1-q)c_n,$$

where c_n is the concentration of He in the n^{th} breath and q is the fraction of air exchanged. Use the data for c_0 and c_1 to estimate the value of q. Then use this model to estimate the concentration of He in the 4^{th} breath (c_4). Determine how many breaths it takes for the He concentration to fall to one half (275 ppm) the original concentration.

$c_n = 550\,(1-q)^n$ $\qquad \frac{500}{550} = 1-q \Rightarrow q = \frac{1}{11}$

$(1-q)^n = \frac{1}{2}$

$n = \frac{\ln(1/2)}{\ln(1-q)}$

2, 2 $\quad q =$ 0.09091 $\qquad c_4 =$ 375.7

2 $\quad$ Concentration He = 275 ppm when $n =$ 7.2725

b. It is determined that there is Helium in the room. The concentration of Helium in the room, γ, is not known, but assumed to be constant. Below is a table of the patient's first two breaths after resuming normal breathing in the room.

Breath Number	0	1	2
Conc. of He (ppm)	550	500	456

Assume a breathing model of the form:

$$c_{n+1} = (1-q)c_n + q\gamma.$$

Use the data above to find the constants, q and γ, the ambient concentration of Helium. Then determine the concentration of Helium in the next two breaths, c_3 and c_4. Assuming that this is the better model, find the percent error between the model in Part a and this model for the estimate of c_4.

$500 = (1-q)\,550 + q\gamma$
$456 = (1-q)\,500 + q\gamma$
$44 = (1-q)\,50$

$1-q = 0.88$

$q\gamma = 500 - 0.88(550) = 16$

$\gamma = 16/q$

$\frac{100\,(375.7 - 383.2)}{383.2}$

3, 3 $\quad q =$ 0.12 $\qquad \gamma =$ 133.3

2, 1, 2 $\quad c_3 =$ 417.28 and $c_4 =$ 383.2 $\qquad$ % Error at $c_4 =$ −1.97 %

c. Find the equilibrium concentration of Helium in the subject's lungs based on the breathing model in Part b. What is the stability of this equilibrium concentration?

$c_e = (1-q)\,c_e + q\gamma \Rightarrow c_e = \gamma$

2, 1 $\quad$ Equilibrium $c_e =$ 133.3 $\qquad$ STABLE or UNSTABLE (Circle one)

7. (21pts) Many ecological studies require that the subject studied is correlated with the temperature of the environment (especially insects and plants). Over a 20 hour period, data are collected on the temperature, $T(t)$in degrees Celsius. The temperature data are found to best fit the cubic polynomial

$$T(t) = 0.01(1600 - 135t + 27t^2 - t^3),$$

where t is in hours (valid for $0 \le t \le 20$).

a. Find the rate of change in temperature per hour, $\frac{dT}{dt}$. What is the rate of change in the temperature at 3 AM, $t = 3$? Also, compute $T''(t)$. When is the rate of change in the temperature increasing the most and what is that maximum rate of increase?

$T'(t) = 0.01(-135 + 54t - 3t^2) = -0.03(t^2 - 18t + 45) = -0.03(t-3)(t-15)$

3, 1 $T'(t) =$ $-1.35 + 0.54t - 0.03t^2$ $T'(3) =$ 0

3 $T''(t) =$ $0.54 - 0.06t$

2, 1 Rate of maximum increase at $t_{inc} =$ 9 $T'(t_{inc}) =$ 1.08 °C/hr

b. Use the derivative to find when the minimum and maximum temperatures occur. Give the temperatures at those times. (11)

3, 1 $t_{max} =$ 15 $T(t_{max}) =$ 22.75 °C

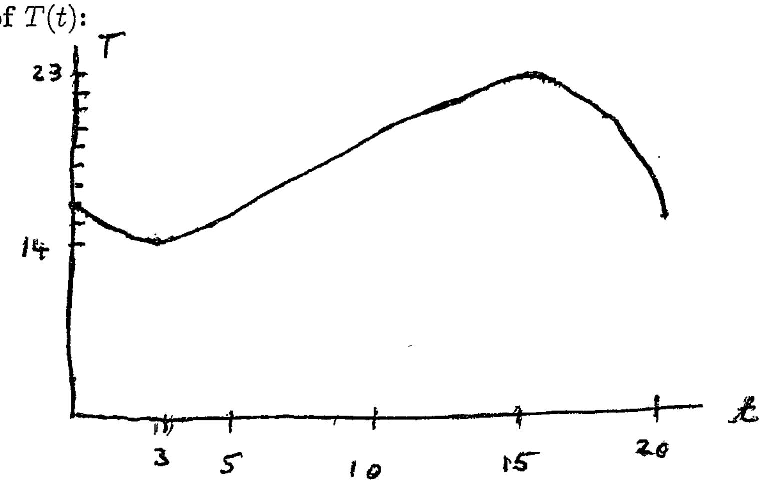
2, 1 $t_{min} =$ 3 $T(t_{min}) =$ 14.11 °C

c. Sketch a graph of this polynomial fit to the temperature. Show clearly the maximum and minimum temperatures on your graph and include the temperatures at the beginning of the study ($t = 0$) and at the end ($t = 20$).

1, 1 $T(0) =$ 16 $T(20) =$ 17

Graph of $T(t)$:

2



8. (22pts) a. The von Bertalanffy equation for growth of fish can be used to approximate the weight of a person. Assume that the weight of a woman, W, in kg as a function of age, t, satisfies the equation:

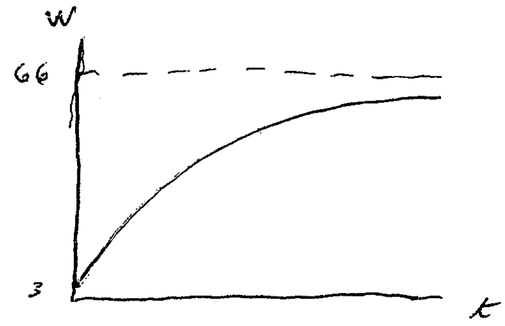
$$W(t) = 66 - 63e^{-0.075t}.$$

Find the age of a woman that the model predicts to weigh 50 kg. Sketch a graph of W showing the W-intercept and the horizontal asymptote.

Graph 2

2 $W(t) = 50$ when $t =$ 18.274

Graph of $W(t)$:



1 $W(0) =$ 3

2 Horizontal Asymptote at $W =$ 66

$50 = 66 - 63e^{-0.075t}$

$e^{0.075t} = 63/16$

$t = \ln(63/16)/0.075$

b. Find the derivative $W'(t)$. Determine the annual weight change for a 10 year old.

2, 1 $W'(t) =$ $4.725e^{-0.075t}$ $W'(10) =$ 2.2319 kg/yr

c. Young children in some urban environments receive a high exposure of lead (Pb). Young children through crawling and oral explorations have the highest lead concentrations. Suppose that that the concentration of lead, c, in a child's body (μg/dl of blood) satisfies

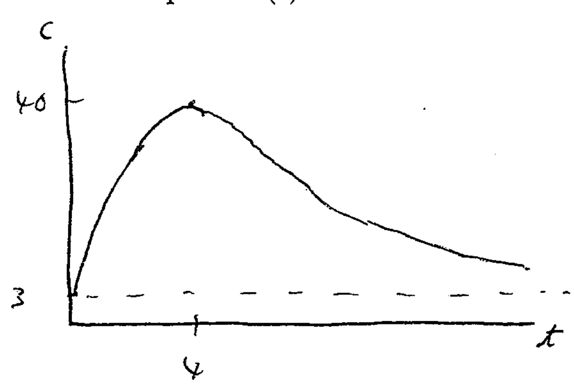
$$c(t) = 3 + 25t\,e^{-0.25t},$$

$c'(t) = 25(t\,e^{-0.25t}(-0.25) + e^{-0.25t})$

where t is the age of the child. Find the derivative $c'(t)$. Find when lead achieves its maximum concentration in the child and determine what its maximum concentration is. Sketch a graph of c showing the c-intercept, the maximum, and any horizontal asymptotes.

3 $c'(t) =$ $25e^{-0.25t}(1 - 0.25t)$

Graph of $c(t)$:



3, 1 $t_{max} =$ 4 $c(t_{max}) =$ 39.79

1 $c(0) =$ 3

2 Horizontal Asymptote at $c =$ 3

Graph 2

$c'(t) = 0 \Rightarrow 1 - 0.25t = 0$

$\Rightarrow t = 4$

$c(4) = 3 + 25(4)e^{-0.25(4)} = 3 + 100e^{-1}$

9. (27pts) Hassell's model has been used to study the population of insects. Let P_n be the population of an agricultural pest in a survey region of 1 m^2 with n in weeks. Suppose that the insect population satisfies the model given by

$$P_{n+1} = H(P_n) = \frac{15\,P_n}{(1+0.01\,P_n)^3}.$$

a. Assume that the initial population is $P_0 = 50$, then determine the population of the pest for the next two weeks (P_1 and P_2).

1, 1 $P_1 =$ 222.2 $/m^2$ $P_2 =$ 99.64 $/m^2$

b. Find $H'(P)$, then determine the maximum of this function (both P and $H(P)$ values). Sketch a graph of $H(P)$ with the identity function for $P \geq 0$, showing the intercepts and any horizontal asymptotes.

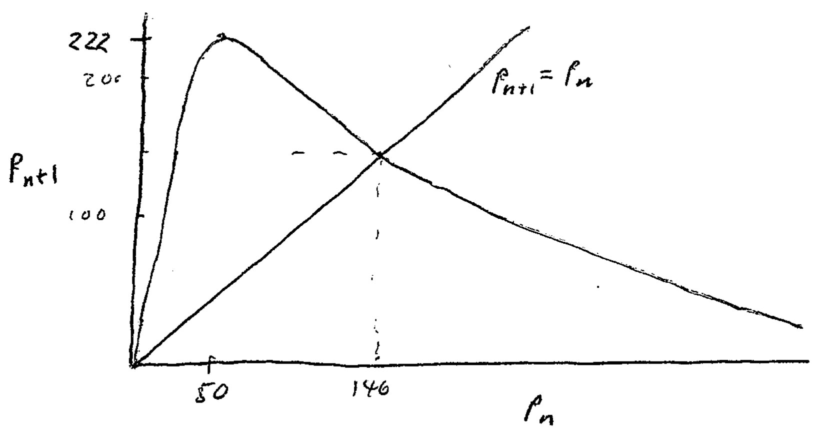
4 $H'(P) = 15\,\dfrac{(1+0.01P)^3 - P\cdot 3(1+0.01P)^2(0.01)}{(1+0.01P)^6} = 15\,\dfrac{(1-0.02P)}{(1+0.01P)^4}$

1, 1, 1 P-intercept 0 H-intercept 0 Horizontal Asymptote $R =$ 0

3, 1 $P_{max} =$ 50 $H(P_{max}) =$ 222.2

GRAPH:

2



c. Find all equilibria for Hassell's model and determine the stability of the equilibria. Justify your stability argument by evaluating the derivative of the updating function. (12)

$P_e = \dfrac{15\,P_e}{(1+0.01P_e)^3}$, $P_e = 0$

$1 + 0.01P_e = 15^{1/3}$

$P_e = 100(15^{1/3} - 1) = 146.62$

$15\,\dfrac{(1-0.02(146.62))}{(1+0.01(146.62))^4}$

1, 2 $P_{1e} =$ 0 $H'(P_{1e}) =$ 15

1, 1 Stable or Unstable Monotonic or Oscillatory

3, 2 $P_{2e} =$ 146.62 $H'(P_{2e}) =$ −0.784

1, 1 Stable or Unstable Monotonic or Oscillatory