CHAPTER 2:

TRIGONOMETRY

1 TRIGONOMETRIC FUNCTIONS

Many phenomena in biology appear in cycles. Often these cycles are driven by the natural physical cycles that result from the daily cycle of light or the annual cycle of the seasons. Oscillations are most easily studied using trigonometric functions. This section begins with a discussion of annual temperature variations, then we review some trigonometric functions. The emphasis for this section is modeling oscillatory behavior with the sine and cosine functions.

1.1 ANNUAL TEMPERATURE CYCLES

Month	Jan	Feb	Mar	Apr	May	Jun
San Diego	66/49	67/51	66/53	68/56	69/59	72/62
Chicago	29/13	34/17	46/29	59/39	70/48	80/58
Month	Jul	Aug	Sep	Oct	Nov	Dec
San Diego	76/66	78/68	77/66	75/61	70/54	66/49
Chicago	84/63	82/62	75/54	63/42	48/32	34/19

Table 1: Table of high and low temperatures of San Diego and Chicago cities through out the year.

Often the weather report states what the average expected temperature of a given day is. These averages are derived from long term collection of data on weather for a particular location[1]. Clearly, there is a wide variation from these averages, but they provide approximations to the expected weather for a particular time of year. The long term averages also provide a baseline to help researchers predict the effects of global warming over the background noise of annual variation. Obviously, there are seasonal differences in the average daily temperature with higher averages in the summer

[1] www.randmcnally.com/, last visited 10/21/09.

and lower averages in the winter. We provide Table 1 showing the monthly average high and low temperatures for San Diego and Chicago.

What mathematical tools can help predict the annual temperature cycles? Polynomials and exponentials do not exhibit the periodic behavior that we see for these average monthly temperatures, so these functions are not appropriate for modeling this system. The most natural candidates for studying monthly temperatures are the trigonometric functions. In Figure 1 are graphs of the average monthly temperatures for San Diego and Chicago, which are computed from Table 1 by averaging the average high and low temperatures.

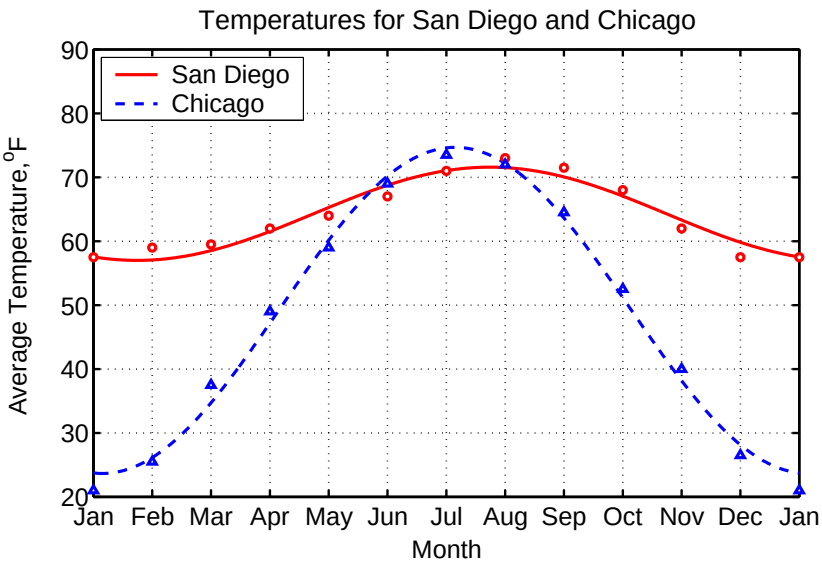

Figure 1: Graph of the monthly average temperature data taken from Table 1, with the curves that best fit the data.

The two graphs of Figure 1 have some similarities and clear differences. They both show the same seasonal period as expected; however, the seasonal variation or amplitude of oscillation for Chicago is much greater than San Diego. Also, the overall average temperature for San Diego, being further south and near the ocean, is greater than the average for Chicago. The overlying models in the graph of Figure 1 use cosine functions. The fit using the cosine function provides a reasonable approximation though clearly there are errors due to other complicating factors in weather prediction. Before providing more details of the models for these temperature cycles, we review some basic facts about trigonometric functions.

1.2 TRIGONOMETRIC FUNCTIONS

The *trigonometric functions* are often called *circular functions*, which emphasizes their periodic nature and shows their connection to a circle. Let (x, y) be a point on a circle of radius r centered at the origin. Define the angle θ between the ray connecting the point to the origin and the x-axis. (See the diagram in Figure 2.)

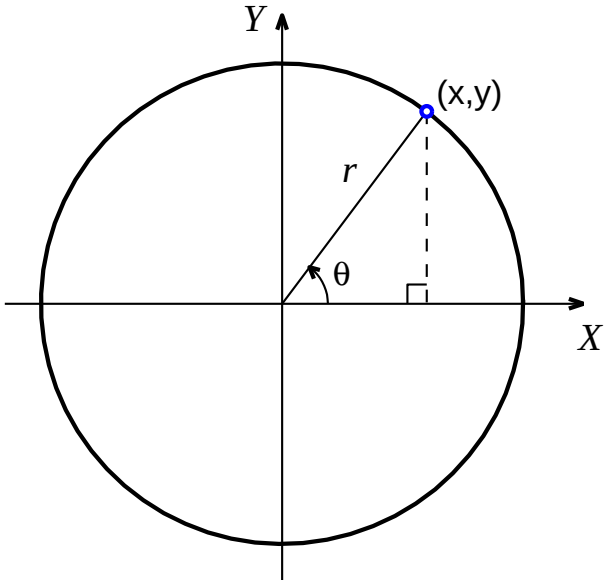

Figure 2: Diagram of a circle on the *Cartesian plane* showing the relationship between *trigonometric functions* and the circle.

The six basic trigonometric functions are defined in terms of x, y, and r (including the sign of x and y) shown in the diagram of Figure 2 by the following,

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x}, \tag{2.1}$$

$$\csc\theta = \frac{r}{y}, \qquad \sec\theta = \frac{r}{x}, \qquad \cot\theta = \frac{x}{y}. \tag{2.2}$$

We will concentrate almost exclusively on the first two of these trigonometric functions, *sine* and *cosine*.

1.3 RADIAN MEASURE

Before discussing the nature of the trigonometric functions in more detail, we need to discuss *radian measure of the angle*. If you have had trigonometry before, you probably used degrees to measure an angle. However, they are *not* the appropriate unit to use in Calculus. The easiest way to consider radian measures is to examine the *unit circle* (which is simply the circle in the diagram in Figure 2 with a radius of 1). From earlier courses, you may recall that the circumference of a circle is $2\pi r$, so the distance around the perimeter of the unit circle is 2π. The radian measure of the angle θ in the diagram of Figure 2 is simply the distance along the circumference of the unit circle. Thus, a $45°$ angle is $\frac{1}{8}$ of the distance around the unit circle or $\frac{2\pi}{8} = \frac{\pi}{4}$ radians. Similarly, $90°$ and $180°$ angles convert to $\frac{\pi}{2}$ and π radians, respectively.

The formulae for converting from degrees to radians or radians to degrees are,

$$1° = \frac{\pi}{180} = 0.01745 \text{ radians},$$

or,

$$1 \text{ radian} = \frac{180°}{\pi} \simeq 57.296°.$$

JAVA *Trigonometric – Degree-Radian Conversion*

This JavaScript is provided for easy conversions of degrees to radians or radians to degrees.

1.4 SINE AND COSINE

From the formulae for sine (sin) and cosine (cos) of Equation (2.1), we see that if you take a unit circle, then the cosine function gives the x value of the angle (measured in radians), while the sine function gives the y value of the angle. The tangent function (tan) gives the slope of the line (y/x). In Figure 3 we have a graph of the sine and cosine functions for the angles from -2π to 2π, *i.e.*, the graph shows $\sin(\theta)$ and $\cos(\theta)$ for $\theta \in [-2\pi, 2\pi]$.

There are several things to notice about these graphs. First, you notice the 2π periodicity. In other words, the functions repeat the same pattern every 2π radians. This is clear from the circle of Figure 2 because every time you go 2π radians around the circle, you return to the same point. The

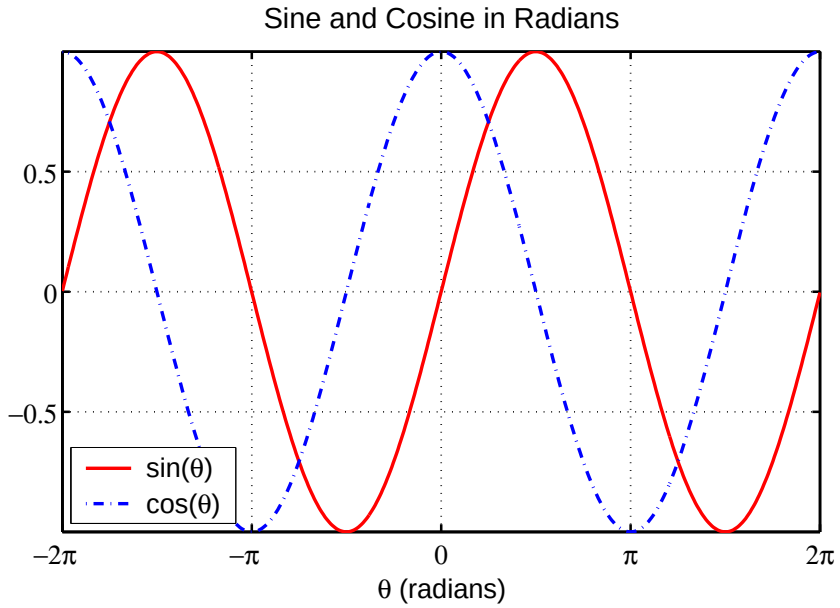

Figure 3: Sine and Cosine functions for radians.

second point is that both the sine and cosine functions are bounded between -1 and 1. The sine function has its maximum value at $\frac{\pi}{2}$ with $\sin\left(\frac{\pi}{2}\right) = 1$ and its minimum value at $\frac{3\pi}{2}$ with $\sin\left(\frac{3\pi}{2}\right) = -1$. Because of the periodicity, these extrema re-occur every integer multiple of 2π.

Trigonometric – Sine and Cosine

This applet helps you link the picture of the circle of Figure 2 and the trigonometric functions, sine and cosine graphed in Figure 3, using the dynamics of an applet. Click on the applet to place the point. Table 2 gives the angle in degrees and radians and values of the sine and cosine functions at the angle chosen.

Table 2 has some important values of the trig functions to remember. The following example considers basic conversions from degrees to radians and finding the values of the sine and cosine functions at certain angles.

θ	$\sin(\theta)$	$\cos(\theta)$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1

Table 2: Some of the most common values of trigonometric functions.

Example 1 *Radians, Sine, and Cosine*

Do not use a calculator for these.

a. Find the radian measure (x) for the following angles given in degrees,

$$\theta = 0^\circ, 30^\circ, 90^\circ, 135^\circ, 210^\circ, 240^\circ, 270^\circ, 315^\circ. \tag{2.3}$$

b. Determine the value of both $\sin(x)$ and $\cos(x)$ for each of these angles.

Solution: a. Recall the conversion formula in the main lecture section. It states that $1^\circ = \pi/180$ radians. Thus, given an angle in degrees θ the radian measure x is given by the formula

$$x = \frac{\theta\pi}{180}.$$

The answers become fractional multiples of π. For example, when $\theta = 135^\circ$, then the radian measure is

$$x = \frac{135\pi}{180} = \frac{3\pi}{4}.$$

The list of the angles in Equation (2.3) can be easily converted as shown in Section 1.3 to yield the following for all of the angles θ listed at (2.3),

$$x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \quad \text{and} \quad \frac{7\pi}{4}.$$

See Table 3 for a more complete listing.

b. To find the sine and cosine values for these angles without the use of a calculator, we refer to Table 2 and use the geometry from the circle. If the angle is located in any quadrant other than the first quadrant, then we need to find the *reference angle*. The *reference angle* is an angle between 0 and $\frac{\pi}{2}$ that is made with the x-axis, as we have θ in Figure 2. Then, for the angle $\frac{7\pi}{6}$ that lies in the third quadrant, we find there is an angle of $\frac{\pi}{6}$ radians between this ray and the negative side of the x-axis. Hence, considering that angle, we get a reference angle of $\frac{\pi}{6}$ radians. The reference angle then gives the magnitude of the sine and cosine functions, we just have to refer to Table 2.

Lastly, we need to assign a sign to the function based on which quadrant it lies. Both sine and cosine are positive in the first quadrant. The sine is positive and the cosine is negative in the second quadrant, while both are negative in the third quadrant. In the fourth quadrant, the cosine is positive and the sine is negative.

The angles that lie on either the x or y-axes are special ones that you should get to know very well to make it much easier to sketch graphs of the trig functions. The angles $x = 0$, $\frac{\pi}{2}$, and $\frac{3\pi}{2}$ are special angles on the axes, while the others (except $x = \frac{\pi}{6}$) require finding the reference angle and which quadrant they reside. We see that when $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$, the reference angle is $\frac{\pi}{4}$, yielding the magnitude of both sine and cosine functions as from the table. However, the first of these is in the second quadrant, while the second is in the third quadrant. Since we discussed the angle $\frac{7\pi}{6}$, the remaining angle is $x = \frac{4\pi}{3}$. It has a reference angle of $\frac{\pi}{3}$ and resides in the third quadrant.

In Table 3 we present a table summarizing our results. $\triangleleft$

degrees (θ)	$0°$	$30°$	$90°$	$135°$	$210°$	$240°$	$270°$	$315°$
radians (x)	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$\sin(x)$	0	$\frac{1}{2}$	1	$\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{2}}{2}$
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{\sqrt{2}}{2}$

Table 3: Results of Example 1.

1.5 PROPERTIES AND IDENTITIES FOR SINE AND COSINE

There are a number of properties that are significant for sine and cosine. Below we list some of the most important properties for cosine.

Properties of Cosine

1. Periodic with period 2π
2. Cosine is an **even** function
3. Cosine is bounded by -1 and 1
4. Maximum at $x = 0$ with $\cos(0) = 1$
   - By periodicity, other maxima at $x_n = 2n\pi$ with $\cos(2n\pi) = 1$ (n any integer)
5. Minimum at $x = \pi$ with $\cos(\pi) = -1$
   - By periodicity, other minima at $x_n = (2n+1)\pi$ with $\cos((2n+1)\pi) = -1$ (n any integer)
6. Zeroes of cosine separated by π with $\cos(x_n) = 0$ when $x_n = \frac{\pi}{2} + n\pi$ (n any integer)

There are a similar set of properties for the sine function.

Properties of Sine

1. Periodic with period 2π
2. Sine is an **odd** function
3. Sine is bounded by -1 and 1
4. Maximum at $x = \frac{\pi}{2}$ with $\sin\left(\frac{\pi}{2}\right) = 1$
   - By periodicity, other maxima at $x_n = 2n\pi + \frac{\pi}{2}$ with $\sin(x_n) = 1$ (n any integer)
5. Minimum at $x = \frac{3\pi}{2}$ with $\sin\left(\frac{3\pi}{2}\right) = -1$
   - By periodicity, other minima at $x_n = 2n\pi + \frac{3\pi}{2}$ with $\sin(x_n) = -1$ (n any integer)
6. Zeroes of sine separated by π with $\sin(x_n) = 0$ when $x_n = n\pi$ (n any integer)

When studying trigonometry, there are many special trigonometric identities that are frequently learned. We are going to highlight a very few that are particularly useful.

Some Identities for Sine and Cosine

1. $\cos^2(x) + \sin^2(x) = 1$ for all values of x
2. Adding and subtracting angles for cosine

$$\begin{aligned} \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y) \end{aligned}$$

3. Adding and subtracting angles for sine

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y) \end{aligned}$$

The first of these identities is readily verified using Pythagorean's Theorem and the definitions of sine and cosine. The second two identities are not as easily shown, and we will omit their proofs here[2]. The second two identities are useful in showing that from a modeling perspective, the sine and cosine functions are often interchangeable.

Example 2 *Sine and Cosine Relationship*

Use the trigonometric properties and identities to show that

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

and

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$$

Solution: We begin using the additive identity formula for sine functions

$$\sin\left(x + \frac{\pi}{2}\right) = \sin(x)\cos\left(\frac{\pi}{2}\right) + \cos(x)\sin\left(\frac{\pi}{2}\right)$$

[2]Proofs use basic definitions of the sine and cosine functions http://en.wikipedia.org/wiki/Proofs_of_trigonometric_identities#Angle_sum_identities - last visited 8/11

By our properties above, we know $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$, so

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x).$$

This relationship says that the cosine function is exactly the same as the sine function, but shifted out of **phase** by $-\frac{\pi}{2}$ (or the sine curve shifted to the left by $\frac{\pi}{2}$).

Using the subtractive identity formula for cosine functions

$$\cos\left(x - \frac{\pi}{2}\right) = \cos(x)\cos\left(\frac{\pi}{2}\right) + \sin(x)\sin\left(\frac{\pi}{2}\right)$$

Again $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$, so

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x).$$

This relationship shows that the sine function has the same form as the cosine function, but shifted out of **phase** by $\frac{\pi}{2}$ (or the cosine curve shifted to the right by $\frac{\pi}{2}$). $\triangleleft$

1.6 PERIOD, AMPLITUDE, PHASE, AND VERTICAL SHIFT

The sine and cosine functions above have a period of 2π and an amplitude of *one*, so how can we adjust these functions to fit other periodic data, such as the temperature data for Chicago and San Diego given in the introduction to this section? When data follows a simple oscillatory behavior, then a general modeling form using the cosine function is

$$y(x) = A + B\,\cos(\omega(x - \phi)), \tag{2.4}$$

while a closely related model using the sine function is

$$y(x) = A + B\,\sin(\omega(x - \phi)). \tag{2.5}$$

There are four parameters in these models. The first parameter is A, which gives the *vertical shift* of the models. The parameter B gives the *amplitude* of the oscillations in the models.

Trigonometric Model Parameters
Vertical Shift and Amplitude

- The model parameter A is the *vertical shift*, which is associated with the average height of the model
- The model parameter B gives the *amplitude*, which measures the distance from the average, A, to the maximum (or minimum) of the model

The parameter ω is the *frequency* of the models.

Trigonometric Model Parameters
Frequency and Period

- The model parameter ω is the *frequency*, which gives the number of periods of the model that occur as x varies over 2π radians
- The *period* is given by $T = \dfrac{2\pi}{\omega}$

It is the periodic nature of the models for which these functions have been chosen. The last parameter, ϕ, is the *phase shift*.

Trigonometric Model Parameters
Phase Shift

- The model parameter ϕ is the *phase shift*, which shifts our models to the left or right, thus, gives a *horizontal shift*
- If the period is denoted $T = \frac{2\pi}{\omega}$, then the *principle phase shift* satisfies $\phi \in [0, T)$
- By periodicity of the model, if ϕ is any *phase shift*, then

$$\phi_1 = \phi + nT = \phi + \frac{2n\pi}{\omega}, \qquad n \text{ an integer}$$

  is a *phase shift* for an equivalent model

When fitting the cosine and sine models, (2.4) and (2.5), there are choices that can be made with the parameters. The parameter, A, is unique. The parameters for amplitude and frequency, B and ω, are unique in magnitude, but the sign of the parameter can be chosen by the modeler. Usually, we prefer to take the parameters B and ω to be positive. Because of the periodicity of these trigonometric functions, there are infinitely many possible choices for the phase shift, ϕ. However, it is customary to select the phase shift satisfying $0 \leq \phi < T$, the *principle phase shift*. By taking the principle phase shift and positive values for the amplitude and frequency, then the four parameters in the cosine and sine models, A, B, ω, and ϕ, are unique.

Example 3 *Period and Amplitude for the Sine Function*

This first example examines amplitude and frequency in the sine model. Find the period and amplitude of

$$y(x) = 4\sin(2x).$$

Determine all maxima and minima for $x \in [-2\pi, 2\pi]$ and sketch the graph.

Solution: From the information above, we see that the amplitude of this function is 4, which says that it oscillates between -4 and 4, and the frequency is 2. The easiest way to find the period T is to let $x = T$ in the function, then set the argument of the trig function to 2π, and finally solve this for T. This means,

$$2T = 2\pi, \qquad \text{so} \qquad T = \pi.$$

Alternately, the definition of period given the frequency, ω, gives:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi.$$

From Table 2, we see that this function begins at 0 when $x = 0$. It achieves a maximum of 4 where the argument $2x = \frac{\pi}{2}$, which gives $x = \frac{\pi}{4}$. The function decreases to a minimum of -4 at $x = \frac{3\pi}{4}$, since the argument there is $2x = \frac{3\pi}{2}$. Then it increases to where it completes its cycle at $x = \pi$. By periodicity, there are other maxima at $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$, $x = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$, and $x = -\frac{9\pi}{4}$. Similarly, there are other minima at $x = -\frac{5\pi}{4}, -\frac{\pi}{4}$, and $\frac{7\pi}{4}$.

The sine function is an odd function, so the graph is symmetric about the origin. The graph is shown in Figure 4 for $x \in [-2\pi, 2\pi]$. ◁

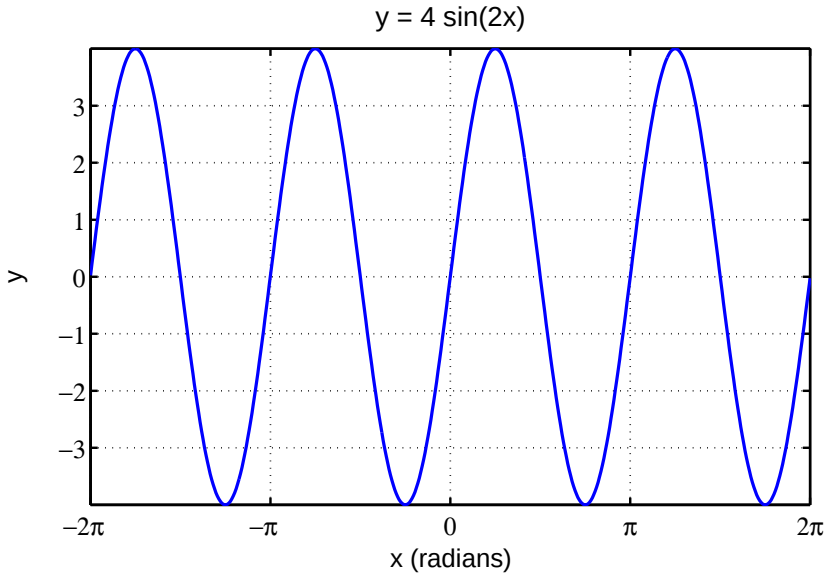

Figure 4: Graph of the function $y = 4\sin(2x)$ of Example 3.

Example 4 *Graphing a Sine Function*

Sketch the graph of the following function,

$$y(x) = 3\sin(2x) - 2,$$

for $-2\pi \leq x \leq 2\pi$. Determine the amplitude and period of oscillation for this function. Find all maxima and minima.

Solution: From above, this function is shifted vertically downward by the constant -2. The amplitude is given by 3 (multiplying the sine function), so the graph will oscillate between $-5 \leq y \leq 1$. These high and low points of the graph are found by letting the sine function take its maximum and minimum values given by 1 and -1, respectively. According to Table 2, this happens when the argument $2x = \frac{\pi}{2}$, which implies $x = \frac{\pi}{4}$. Then $\sin(2x) = \sin(\pi/2) = 1$. Thus, at $x = \frac{\pi}{4}$,

$$y(\pi/4) = 3(1) - 2 = 1.$$

When $2x = \frac{3\pi}{2}$, or $x = \frac{3\pi}{4}$, then $\sin(2x) = \sin(3\pi/2) = -1$. Thus, at $x = 3\pi/4$ we have,

$$y(3\pi/4) = 3(-1) - 2 = -5.$$

To find the period, T, we have

$$T = \frac{2\pi}{\omega}, \qquad \text{so} \qquad T = \frac{2\pi}{2} = \pi.$$

Thus, the period of this function is π.

The best way to sketch a graph of either the sine and cosine function is to take the period of the function, then divide it into 4 even parts. For this example, we divide the interval $[0, \pi]$ into 4 parts. Next we evaluate the function at each of the endpoints of these subintervals, which for this case occurs at $0, \pi/4, \pi/2, 3\pi/4$, and π. We obtain the following,

$$\begin{aligned}
y(0) &= 3\sin(2(0)) - 2 = 3\sin(0) - 2 = -2, \\
y(\pi/4) &= 3\sin(2(\pi/4)) - 2 = 3\sin(\pi/2) - 2 = 1, \\
y(\pi/2) &= 3\sin(2(\pi/2)) - 2 = 3\sin(\pi) - 2 = -2, \\
y(3\pi/4) &= 3\sin(2(3\pi/4)) - 2 = 3\sin(3\pi/2) - 2 = -5, \\
y(\pi) &= 3\sin(2(\pi)) - 2 = 3\sin(2\pi) - 2 = -2.
\end{aligned}$$

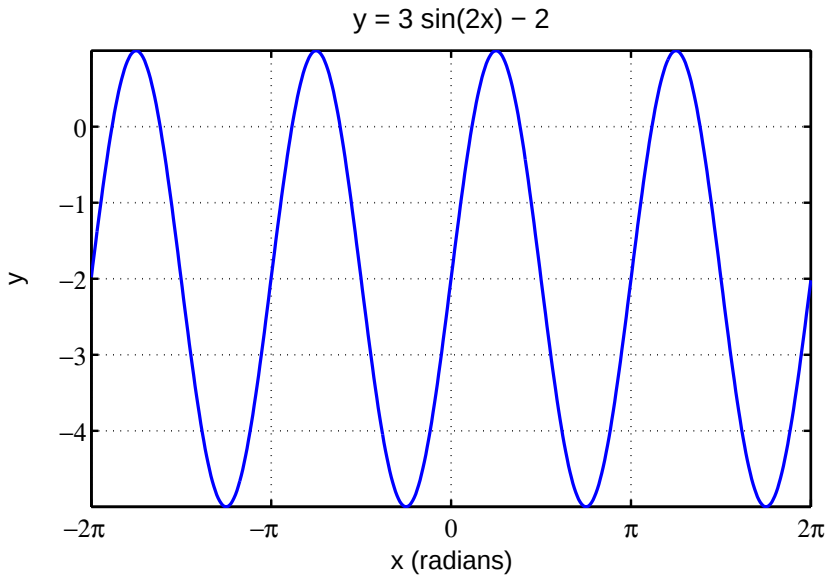

Figure 5: Graph of the function $y(x) = 3\sin(2x) - 2$ of Example 4.

This takes on the important values (minima and maxima) and goes through one cycle, which makes the sketching easy. One simply repeats the graph to extend it. In Figure 5 there is a graph of this function.

Since this function is periodic with period π and a maximum of $y = 1$ occurs at $x = \frac{\pi}{4}$, there are the other maxima at $x = \frac{\pi}{4} - 2\pi = -\frac{7\pi}{4}$, $\frac{\pi}{4} - \pi = -\frac{3\pi}{4}$, and $\frac{\pi}{4} + \pi = \frac{5\pi}{4}$. Since a minimum of $y = -5$ occurs at $x = \frac{3\pi}{4}$, we can add and subtract integer multiples of π to obtain the other minima at $x = -\frac{5\pi}{4}$, $-\frac{\pi}{4}$, and $\frac{7\pi}{4}$. ◁

Example 5 *Vertical Shift with the Cosine Function*

Consider a model given by

$$y(x) = 3 - 2\cos(3x), \qquad x \in [0, 2\pi].$$

Find the *vertical shift*, *period*, and *amplitude* of this model. Determine all minima and maxima in the domain and sketch the graph.

Solution: This function is shifted vertically by the constant 3. The amplitude is the 2 multiplying the cosine function, but there is a negative sign in this model, which forces the model in the opposite direction of the cosine model given by (2.4). The vertical shift of 3 and amplitude of 2 means that the graph oscillates between $1 \le y \le 5$, since the range of the cosine function varies between -1 and 1.

To find the period, T, we solve

$$3T = 2\pi, \qquad \text{so} \qquad T = \frac{2\pi}{3}.$$

Thus, the period of this function is $\frac{2\pi}{3}$.

The easiest way to find a minimum or maximum is to use the largest and smallest values of the cosine function, 1 and -1, which occur when the argument is 0 and π, see Table 2. Hence, when $x = 0$, $\cos(3x) = 1$ and

$$y(0) = 3 - 2(1) = 1,$$

which is a minimum. When $x = \frac{\pi}{3}$, $\cos(3x) = \cos(\pi) = -1$ and

$$y(\pi/3) = 3 - 2(-1) = 5,$$

which is a maximum. By periodicity, the other minima occur at integer multiples of $\frac{2\pi}{3}$, which gives the minima in the domain at $x = 0$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, and 2π. Similarly, the maxima in the domain occur at $x = \frac{\pi}{3}$, π, and $\frac{5\pi}{3}$. Figure 6 is a graph of this function.

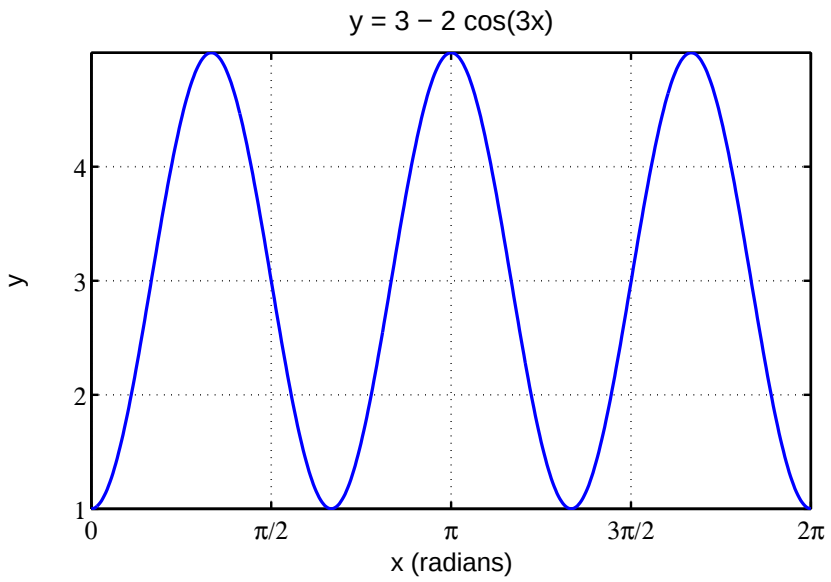

Figure 6: Graph of the function $y = 3 - 2\cos(3x)$ of Example 5.

We can insert a phase shift of half a period to make the constant for the amplitude be positive and produce the same model. Consider the model with a half period phase shift given by

$$y(x) = 3 + 2\cos\left(3(x - \tfrac{\pi}{3})\right).$$

To show this is the same model we employ the angle subtraction identity for the cosine function.

$$\begin{aligned} y(x) &= 3 + 2\cos\left(3(x - \tfrac{\pi}{3})\right), \\ &= 3 + 2\cos(3x - \pi), \\ &= 3 + 2(\cos(3x)\cos(\pi) + \sin(3x)\sin(\pi)), \\ &= 3 - 2\cos(3x), \end{aligned}$$

since $\cos(\pi) = -1$ and $\sin(\pi) = 0$. ◁

Phase Shift of Half a Period

A *phase shift* of *half a period* creates an equivalent sine or cosine model with the sign of the amplitude reversed.

Phase shifts are important matching data in periodic models. The easiest model to match a phase shift is the cosine model (2.4), since the maximum of the cosine function occurs when the argument is zero. The next example explores graphing the cosine model with a phase shift and gives the corresponding sine model that produces the same graph.

Example 6 *Graphing a Cosine Function with a Phase Shift*

Consider the following cosine model, which includes a phase shift

$$y(x) = 4 + 6\cos\left(\frac{x-\pi}{2}\right), \qquad x \in [-4\pi, 4\pi].$$

Find the *vertical shift*, *amplitude*, *period*, and *phase shift* for this model. Determine all maxima and minima in the domain. Finally, find the equivalent sine model with the principle phase shift (a phase shift, $\phi \in [0, T)$, where T is the period of the model.

Solution: From the form of the model it is easy to read off the *vertical shift*, given by $A = 4$. The *amplitude* is similarly easy to read as $B = 6$. If we rewrite the function as,

$$y(x) = 4 + 6\cos\left(\tfrac{1}{2}(x-\pi)\right),$$

then the frequency of the model is $\omega = \frac{1}{2}$. This allows computation of the period, T,

$$T = \frac{2\pi}{\frac{1}{2}} = 4\pi.$$

This also allows easy reading of the *phase shift*, $\phi = \pi$.

The phase shift indicates that this is a cosine function shifted horizontally $x = \pi$ units to the right. Since the cosine function has a maximum value when its argument is zero (see Table 2), this model will achieve a maximum at $x = \pi$. With the period being $T = 4\pi$, the easiest way to graph this function is to start at $x = \pi$ and proceed to $x = 5\pi$, completing one period. Then we use the periodicity to get the desired graph over the domain.

The significant points for evaluation are always each quarter of the period. Thus, for this model the points of interest will be $x = \pi$, 2π, 3π, 4π, and 5π. (Note that the last point is outside the domain.) We substitute

these values into the cosine model to complete one period, giving

$$\begin{aligned}
y(\pi) &= 4+6\cos\left(\frac{\pi-\pi}{2}\right) = 4+6\cos(0) = 4+6(1) = 10,\\
y(2\pi) &= 4+6\cos\left(\frac{2\pi-\pi}{2}\right) = 4+6\cos\left(\frac{\pi}{2}\right) = 4+6(0) = 4,\\
y(3\pi) &= 4+6\cos\left(\frac{3\pi-\pi}{2}\right) = 4+6\cos(\pi) = 4+6(-1) = -2,\\
y(4\pi) &= 4+6\cos\left(\frac{4\pi-\pi}{2}\right) = 4+6\cos\left(\frac{3\pi}{2}\right) = 4+6(0) = 4,\\
y(5\pi) &= 4+6\cos\left(\frac{5\pi-\pi}{2}\right) = 4+6\cos(2\pi) = 4+6(1) = 10.
\end{aligned}$$

Since this function has period, $T = 4\pi$, we cycle backwards one and a quarter periods to easily obtain the values of the model with $y(0) = 4$, $y(-\pi) = -2$, $y(-2\pi) = 4$, $y(-3\pi) = 10$, and $y(-4\pi) = 4$,. The graph is readily produced from the function evaluations given on the domain $x \in [-4\pi.4\pi]$. In Figure 7 there is a graph of this function.

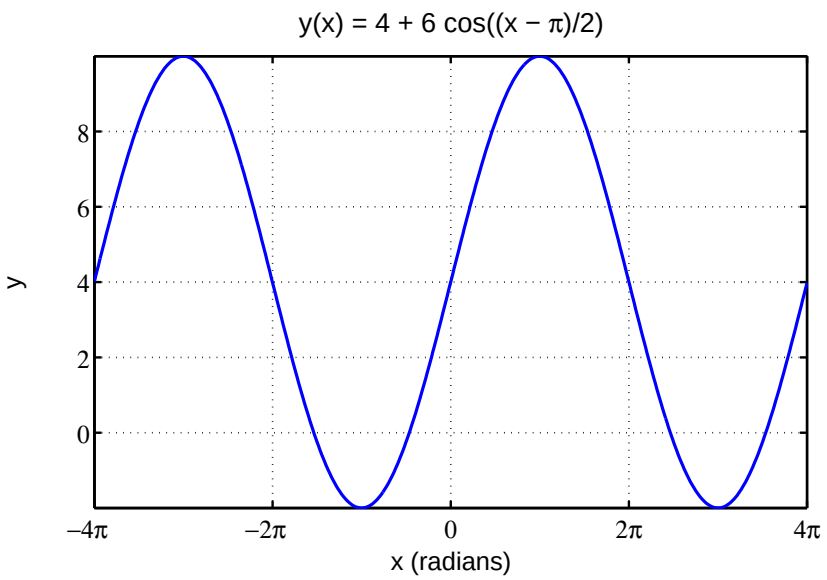

Figure 7: Graph of the function $y(x) = 4 + 6\cos\left(\frac{x-\pi}{2}\right)$ of Example 6.

Observing the graph, we see that the model is vertically shifted by $y = 4$ and oscillates about this line. As noted above, it has an amplitude of $B = 6$,

so oscillates 6 units above and below this average line with a period of $T = 4\pi$. Finally, we note that this cosine function is shifted to the right by the phase shift, $\phi = \pi$.

From either the graph or the values computed above, we see that in the domain, this model has maxima of $y(x_{max}) = 10$ at $x_{max} = -3\pi$ and π. The model has minima of $y(x_{min}) = -2$ at $x_{min} = -\pi$ and 3π.

A closer look at the graph in Figure 7 indicates that this model looks more like a standard sine function model. Suppose we want to use the sine model

$$y(x) = A + B\sin(\omega(x - \phi)).$$

Since we want the graph to be identical, the vertical shift, amplitude, and period must be the same, so $A = 4$, $B = 6$, and $\omega = \frac{1}{2}$ or

$$y(x) = 4 + 6\sin\left(\tfrac{1}{2}(x - \phi)\right).$$

It remains to find the appropriate *phase shift*, ϕ.

Recall from Example 2 that the cosine function is horizontally shifted to the left by $\frac{\pi}{2}$ of the sine function. Thus,

$$\cos\left(\tfrac{1}{2}(x - \pi)\right) = \sin\left(\tfrac{1}{2}(x - \pi) + \tfrac{\pi}{2}\right) = \sin\left(\tfrac{1}{2}(x - \phi)\right).$$

It follows that we want

$$-\frac{\pi}{2} + \frac{\pi}{2} = -\frac{\phi}{2} \qquad \text{or} \qquad \phi = 0.$$

Thus, there is no phase shift for the sine model, and the equivalent sine model is given by

$$y(x) = 4 + 6\sin\left(\tfrac{x}{2}\right).$$

◁

The example above shows that models using sine or cosine functions are equivalent when the *vertical shift*, *amplitude*, and *period* are the same. Only the *phase shift* differs, and our trigonometric identities readily find this difference in the phase shift.

Phase Shift for Equivalent Sine and Cosine Models
Suppose that the sine and cosine models are equivalent, so

$$\sin(\omega(x-\phi_1)) = \cos(\omega(x-\phi_2)).$$

The relationship between the *phase shifts*, ϕ_1 and ϕ_2 satisfies:

$$\phi_1 = \phi_2 - \frac{\pi}{2\omega}.$$

The identity above is easily shown using the result from Example 2. This example showed that

$$\sin(\omega(x-\phi_1)) = \cos\left(\omega(x-\phi_1) - \tfrac{\pi}{2}\right) = \cos(\omega(x-\phi_2)).$$

Equating arguments of the cosine function gives

$$\begin{aligned} \omega(x-\phi_1) - \tfrac{\pi}{2} &= \omega(x-\phi_2), \\ -\omega\phi_1 - \tfrac{\pi}{2} &= -\omega\phi_2, \\ \phi_1 &= \phi_2 - \frac{\pi}{2\omega}. \end{aligned}$$

Note: It is important to remember that the phase shift is not unique and can vary by integer multiples of the period, $T = \frac{2\pi}{\omega}$.

1.7 RETURN TO THE ANNUAL TEMPERATURE VARIATION

At the beginning of this section, there is an example showing the temperature variation between the seasons for Chicago and San Diego. We want to show what the mathematical models are for the curves in the graphs and explain them in terms of the definitions listed above to give you a better intuitive feel for the *period, amplitude*, *phase shift*, and *vertical shift*. The cosine model has the form

$$T(m) = A + B\cos(\omega(m-\phi)),$$

where T is the average monthly temperature and m is the month with January satisfying $m = 0$. We need to find the appropriate values for the parameters A, B, ω, and ϕ, and we want to select $B > 0$, $\omega > 0$, and $\phi \in [0, P)$, where P is the period of the model. The method for fitting the actual data to the model employs a couple of techniques. First, we know

that the period of this function must be 12 months. This constrains our parameter ω to satisfy

$$12\omega = 2\pi \quad \text{or} \quad \omega = \frac{\pi}{6} = 0.5236.$$

We know that the cosine function oscillates around its average, so the value of A, which gives the vertical shift, is the average of the monthly temperatures. From Table 1, we find the average temperature of San Diego for each month by averaging the minimum and maximum temperatures recorded each month. The average temperature for the year satisfies:

$$\begin{aligned} A &= \tfrac{57.5+59+59.5+62+64+67+71+72.5+71.5+68+62+57.5}{12} \\ &= 64.29, \end{aligned}$$

while for Chicago, we find

$$A = 49.17.$$

There are a several methods to obtain fits for the parameters B and ϕ. We chose to find a nonlinear least squares fit to these parameters with Excel's Solver. The model for the average monthly temperature in San Diego is given by

$$T(m) = 64.29 + 7.29\cos(0.5236(m - 6.74)),$$

while the average monthly temperature in Chicago follows the formula

$$T(m) = 49.17 + 25.51\cos(0.5236(m - 6.15)).$$

Here, we have again that T is the average monthly temperature and m is the month number with January satisfying $m = 0$.

We have discussed the *vertical shift* (average temperature given by A) and the *period* (12 months giving a frequency of $\omega = \frac{\pi}{6} = 0.5236$). The *amplitude* is given by the parameter B, which represents the maximum amount the temperature varies from the annual average. With its "Mediterranean" climate, San Diego has the significantly lower amplitude with only a 7.29°F variation higher and lower than its average annual temperature. Chicago has a temperature variation of 25.51°F above and below its lower average of 49.17°F. Thus, our model predicts that the temperature of San Diego will vary from 57.0°F to 71.58°F (average monthly temperature), while Chicago will vary from 23.66°F to 74.68°F (average monthly temperature). These should be apparent from the graph of Figure 1.

Finally, we interpret the *phase shift*, ϕ, which indicates that the graph is shifted ϕ units to the right. For San Diego, we see a phase shift of $\phi = 6.74$.

This means that the maximum temperature occurs at 6.74 months (late July), instead of January, as one would expect. For Chicago, the phase shift is $\phi = 6.15$ months, which gives the high occurring a little earlier (early July). Be sure to view the original graph of Figure 1, and match how the parameters are reflected in this graph as it is important for understanding the use of trigonometric functions.

The cosine models given above for the average temperature of San Diego and Chicago can be rewritten with the sine function,

$$T(m) = A + B\sin(\omega(m - \phi_2)).$$

The values for A, B, and ω are the same for both the sine and cosine models. However, the *phase shift* for the sine model differs from the cosine model. The formula above shows that the sine phase shift, ϕ_2, satisfies

$$\phi_2 = \phi - \frac{\pi}{2\omega},$$

where ϕ is the phase shift from the cosine model. This formula gives $\phi_2 = 3.74$ for San Diego and $\phi_2 = 3.15$ for Chicago. This phase shift for the sine model is a quarter of the period less than the phase shift for the cosine model. Thus, we write the sine models for the average monthly temperature in San Diego,

$$T(m) = 64.29 + 7.29\sin(0.5236(m - 3.74)),$$

and the average monthly temperature in Chicago,

$$T(m) = 49.17 + 25.51\sin(0.5236(m - 3.15)).$$

These models produce exactly the same graphs as seen in Figure 1.

Example 7 *Model with Phase Shift*

Consider an oscillatory set of population data that is periodic with a period of 10 yr. Suppose that there is a maximum population (in thousands) of 26 at $t = 2$ and a minimum population (in thousands) of 14 at $t = 7$. Assume these data fit a model of the form

$$y(t) = A + B\,\sin(\omega(t - \phi)).$$

Find the appropriate constants A, B, ω, and ϕ. Choose $B > 0$ and $\omega > 0$, then find $\phi \in [0, 10)$. Since ϕ is not unique, find values of ϕ with $\phi \in [-10, 0)$

and $\phi \in [10, 20)$. Graph the model. In addition, repeat this process for the cosine model

$$y(t) = A + B\,\cos(\omega(t - \phi_2)).$$

Solution: The vertical shift, A, is the average of the high and low points of the data, so

$$A = \tfrac{26+14}{2} = 20.$$

The amplitude, B, is the distance from the maximum to the average, so

$$B = 26 - 20 = 6.$$

Since the period is $T = 10$ years, the frequency, ω, satisfies

$$\omega = \tfrac{2\pi}{10} = \tfrac{\pi}{5}.$$

Note that the times of the maximum and minimum are separated by half a period. This will always be the case of our sine and cosine models.

We are given that the maximum of 26 occurs at $t = 2$, so the model satisfies:

$$y(2) = 26 = 20 + 6\,\sin\left(\tfrac{\pi}{5}(2 - \phi)\right).$$

From this equation it is clear that

$$\sin\left(\tfrac{\pi}{5}(2 - \phi)\right) = 1.$$

Recall that the sine function is at its maximum with value one when its argument is $\frac{\pi}{2}$, so

$$\begin{aligned} \frac{\pi}{5}(2 - \phi) &= \tfrac{\pi}{2}, \\ 2 - \phi &= \tfrac{5}{2}, \\ \phi &= -\tfrac{1}{2}. \end{aligned}$$

This value of ϕ is not in the interval $[0, 10)$, but the periodicity, $T = 10$, of the model is also reflected in the phase shift, ϕ. We can write

$$\begin{aligned} \phi &= -\tfrac{1}{2} + 10\,n, \qquad n \text{ an integer} \\ \phi &= \ldots - 10.5, -0.5, 9.5, 19.5, \ldots \end{aligned}$$

with the *principle phase shift* being $\phi = 9.5$. It follows that we can write the sine model

$$y(t) = 20 + 6\,\sin\left(\tfrac{\pi}{5}(t - 9.5)\right),$$

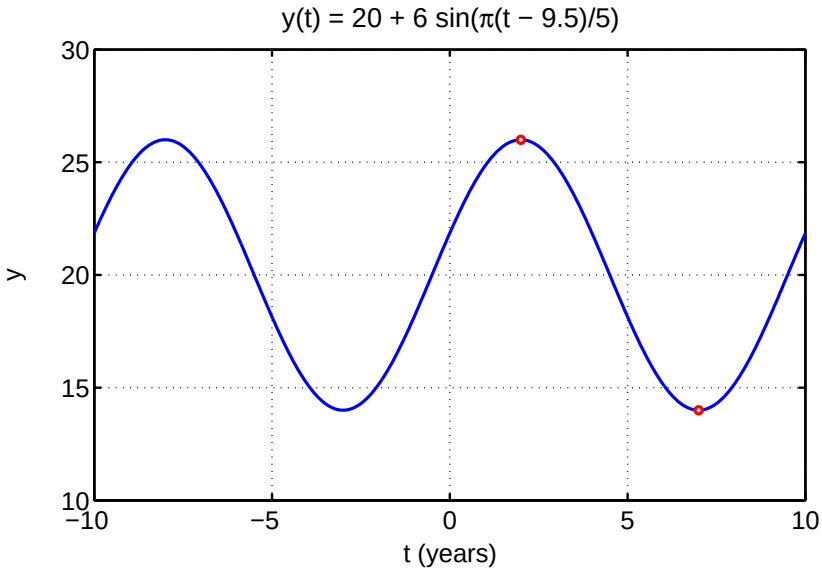

Figure 8: Graph of the model $y(t) = 20 + 6\sin(\pi(t - 9.5)/5)$ of Example 7.

and its graph is given in Figure 8. From the formula for ϕ, we could also use phase shifts of $\phi = -\frac{1}{2}$ or $\phi = 19.5$.

The cosine model has the form

$$y(t) = 20 + 6\,\cos\left(\tfrac{\pi}{5}(t - \phi_2)\right),$$

where the vertical shift, amplitude, and frequency match the sine model. It remains to find the phase shift, ϕ_2. The maximum of the cosine function occurs when its argument is 0, so

$$\begin{aligned} \frac{\pi}{5}(2 - \phi_2) &= 0, \\ \phi_2 &= 2. \end{aligned}$$

It follows that the cosine model satisfies

$$y(t) = 20 + 6\,\cos\left(\tfrac{\pi}{5}(t - 2)\right).$$

By periodicity of the phase shift, we have

$$\begin{aligned} \phi_2 &= 2 + 10\,n, \qquad n \text{ an integer} \\ \phi_2 &= \ldots - 8, 2, 12, 22, \ldots, \end{aligned}$$

where the *principle phase shift* is $\phi_2 = 2$. ◁

1.8 BODY TEMPERATURE

Humans, like many organisms, undergo *circadian rhythms* for many of their bodily functions. Circadian rhythms are the daily fluctuations that are driven by the light/dark cycle of the Earth, which seems to affect the pineal gland in the head. The average body temperature for a human is about 37°C. However, this temperature normally varies a few tenths of a degree in each individual with distinct regularity. The body is usually at its hottest around 10 or 11 AM and at its coolest in the late evening, which helps encourage sleep.

Suppose that measurements on a particular individual show that he has a high body temperature of 37.1°C at 10 am. He has a low body temperature of 36.7°C at 10 pm. Assume that his body temperature, $T(t)$, satisfies the following equation,

$$T(t) = A + B\cos(\omega(t-\phi)), \tag{2.6}$$

for some parameters A, B, ω, and ϕ. Use the data above to find the four parameters with $B > 0$, $\omega > 0$, and $\phi \in [0, 24)$. We show how to select reasonable parameter values for this cosine model. We also create the sine model.

Solution: In this problem, we know that one day is 24 hours, so the period of the function is 24. (Be careful not to confuse the temperature $T(t)$ with our previous notation of T as the period.) The frequency is given by

$$\omega = \frac{2\pi}{24} = \frac{\pi}{12}.$$

As seen in our examples above, the parameter A represents the average temperature of the body. The mean (average) temperature is half-way between the extreme values, so

$$A = \frac{37.1°\text{C} + 36.7°\text{C}}{2} = 36.9°\text{C}.$$

The difference between the maximum and the average temperature gives the amplitude of variation in the body temperature. We can see that the difference from the mean to the maximum (or the minimum) is 0.2°C, which is the amplitude of this trigonometric model, B.

Last, the phase will determine at what time the highest point occurs. The cosine function has its maximum when its argument is 0 (or any integer multiple of 2π). Since the highest body temperature occurs at $t = 10$, we find the appropriate phase shift by solving

$$\omega(10-\phi) = 0 \qquad \text{or} \qquad \phi = 10.$$

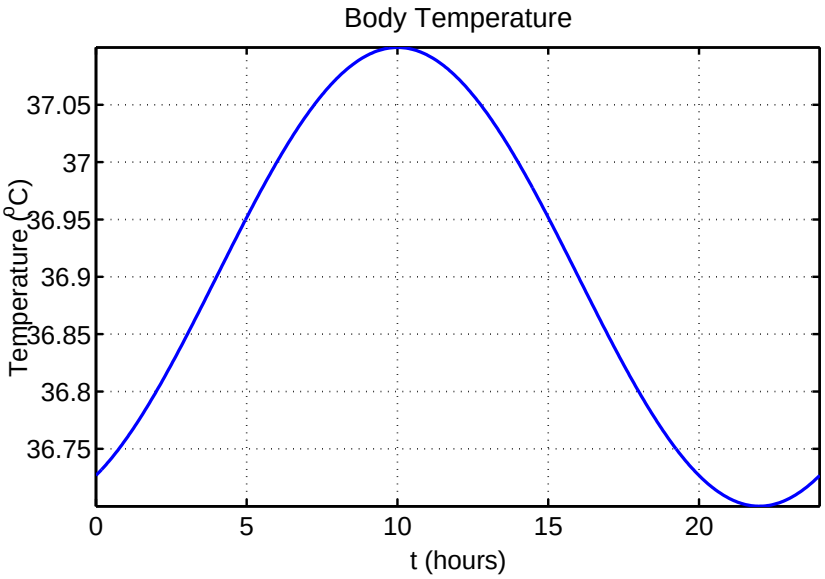

Figure 9: Graph of the body temperature.

It follows that the phase shift is given by $\phi = 10$.

Using these values for the parameters in the Equation (2.6) results in an equation modeling the temperature of the body at a given time,

$$T(t) = 36.9 + 0.2\,\cos\left(\tfrac{\pi}{12}(t-10)\right),$$

which is shown graphically in Figure 9. We see from the graph of Figure 9 that the maximum is 37.1°C, and it is shifted to 10 AM, a time when most of us are becoming active. The minimum occurs at 10 PM. A tendency of this minimum to be delayed in adolescence can result in sleep disorders, abnormal sleep patterns, and chronic sleep deprivation.

If the body temperature is modeled by a sine function of the form

$$T(t) = A + B\,\sin(\omega(t-\phi_2)),$$

then the parameters $A = 36.9$, $B = 0.2$, and $\omega = \frac{\pi}{12}$, the same as for the cosine model. It only remains to find the phase shift, ϕ_2. As seen before, the phase shift for the sine model is a quarter period less than the one for the cosine model. From our formula above,

$$\phi_2 = 10 - \frac{\pi}{2\omega} = 10 - 6 = 4.$$

Thus, the equivalent sine model is given by

$$T(t) = 36.9 + 0.2\,\sin\left(\tfrac{\pi}{12}(t-4)\right).$$

1.9 EXERCISES

1. Complete the following table. Do **NOT** use a calculator! When the value for the sine or cosine function is given, state all possible solutions for the angles with $0 \leq x \leq 2\pi$ or $0 \leq \theta \leq 360°$.

radian (x)	degree (θ)	$\sin(x)$	$\cos(x)$
$\frac{3\pi}{4}$			
		$-\frac{1}{2}$	
	$330°$		
			$-\frac{\sqrt{3}}{2}$
$\frac{10\pi}{3}$			
		$\frac{\sqrt{2}}{2}$	
	$210°$		
			-1
$-\frac{5\pi}{4}$			
		0	
	$270°$		
			$-\frac{1}{2}$

Sketch a graph of the following trigonometric functions for $-2\pi \leq x \leq 2\pi$. Give the period of the function.

2. $y(x) = 3\cos(2x)$
3. $y(x) = 2 - 4\sin(3x)$
4. $y(x) = 1 + 3\cos(2x)$
5. $y(x) = 4\sin\left(\frac{x}{2}\right)$
6. $y(x) = 2\sin(4x) + 1$
7. $y(x) = 5 - 2\cos\left(\frac{x}{2}\right)$
8. $y(x) = 2 - \cos(2(x - \pi))$
9. $y(x) = 2\sin\left(3\left(x + \frac{\pi}{2}\right)\right)$

10. For each of the following problems (see 7 and 8 above) find an equivalent model of the form

$$y(x) = A + B\,\cos(\omega(x - \phi)),$$

where the amplitude, $B > 0$, and the principle phase shift, $\phi \in [0, T)$ with T being the period of the function.

a. $y(x) = 5 - 2\cos\left(\frac{x}{2}\right)$

b. $y(x) = 2 - \cos(2(x - \pi))$

11. For each of the following problems (see 3 and 9 above) find an equivalent model of the form

$$y(x) = A + B\,\sin(\omega(x - \phi)),$$

where the amplitude, $B > 0$, and the principle phase shift, $\phi \in [0, T)$ with T being the period of the function.

a. $y(x) = 2 - 4\sin(3x)$ b. $y(x) = 2\sin\left(3\left(x + \frac{\pi}{2}\right)\right)$

12. The lungs do not completely empty or completely fill in normal breathing. The volume of the lungs normally varies between 2200 ml and 2800 ml with a breathing rate of 24 breaths/min. This exchange of air is called the *tidal volume*. One approximation for the volume of air in the lungs uses the cosine function written in the following manner,

$$V(t) = A + B\cos(\omega t),$$

where A, B, and ω are constants and t is in minutes. Use the data above to create a model, *i.e.*, find A, B, and ω that simulates the normal breathing of an individual for one minute. Graph the function for 10 sec., clearly showing the maximum and minimum volumes, and frequency of inhalation.

13. a. The heart pumps blood at a regular rate of about 60 pulses per minute. The heart volume is about 140 ml, and it pushes out about 1/2 its volume (70 ml) with each beat. Use a model of the following form to simulate the volume of blood, $B(t)$, in the heart at any time t,

$$B(t) = a + b\,\sin(\omega t),$$

where a, b, and ω are constants and t is in minutes. Sketch a graph of this function for 5 sec., clearly showing the maximum and minimum volumes, and frequency of the beating heart.

b. When the heart pushes out blood, the pressure, $P(t)$, in the aorta and arterioles increases to 120 mm Hg. When the heart fills with blood, the pressure falls to about 80 mm Hg. Use a similar model of the form

$$P(t) = c + d\,\sin(\omega t),$$

where c, d, and ω are constants with d and ω positive and t is in minutes. Again you sketch a graph of this function for 5 sec., clearly showing the maximum and minimum volumes, and frequency of the beating heart.

c. Consider your answer in Part b, and determine an equivalent model of the form

$$P(t) = C + D\cos(\nu(t - \phi)),$$

where $D > 0$, $\nu > 0$, and $\phi \in [0, T)$ with T being the period of the heart. Relate your constants in this model to the one in Part b.

14. The average body temperature for a human is about 37°C. However, this temperature normally varies a few tenths of a degree in each individual with distinct regularity. The body is usually at its hottest around 10 or 11 am and at its coolest in the late evening, which helps encourage sleep. When an individual switches to night shift work, his body temperature cycle has to switch also.

a. Suppose that a worker on the night shift finds his hottest body temperature to be at 2 am with a body temperature of 37.1°C, then 12 hours later his body temperature achieves a minimum of 36.7°C. Assume that the body temperature can be modeled using a trigonometric function and is given by

$$T(t) = A + B\cos(\omega(t - \phi)),$$

where A, $B > 0$, $\omega > 0$, and $\phi \in [0, 24)$ are constants and t is in hours. Use the data above to find the four parameters, then sketch a graph for the temperature of this individual for one day.

b. Determine an equivalent temperature model of the form

$$T(t) = C + D\sin(\nu(t - \psi)),$$

where $D > 0$, $\nu > 0$, and $\psi \in [0, 24)$. Relate your constants in this model to the one above. Also, find a value of the phase shift $\psi \in [-24, 0)$, which produces an equivalent model.

15. a. Iguanas are cold-blooded or ectothermic organisms with their body temperature depending on the external temperature. (See www.anapsid.org for more information.) Their natural habitat lies near the equator, where the sun shines about 12 hours a day. The iguana's temperature cycles during the day, with a low of 75°F at about 3 am and a high of 104°F at about 3 pm. Assume that the body temperature of an iguana can be modeled using the following function,

$$T(t) = A + B\sin(\omega(t - \phi)),$$

where A, B, ω, and ϕ are constants and t is in hours. Use the data above to find the four parameters, then sketch a graph for the temperature of a typical iguana for one day.

b. A temperature of 88°F for at least 12 hours a day is critical for the health of an iguana. About how many hours a day does your iguana model give this temperature? (Use the graph which you have created to make a reasonable estimate.)

c. Determine an equivalent temperature model of the form

$$T(t) = C + D\cos(\nu(t - \psi)),$$

where $D > 0$, $\nu > 0$, and $\psi \in [0, 24)$. Relate your constants in this model to the one above. Also, find a value of the phase shift $\psi \in [-24, 0)$, which produces an equivalent model.

16. During the human female menstrual cycle, the gonadotropin, FSH or follicle stimulating hormone, is released from the pituitary in a sinusoidal manner with a period of approximately 28 days. Guyton's text on *Medical Physiology* [1] shows that if we define day 0 ($t = 0$) as the beginning of menstruation, then FSH, $F(t)$, cycles with a high concentration of about 4 ("relative units") around day 9 and a low concentration of about 1.5 around day 23.

a. Consider a model of the concentration FSH (in "relative units") given by,

$$F(t) = A + B\cos(\omega(t - \phi)),$$

where A, B, ω, and ϕ are constants and t is in days. Use the data above to find the four parameters, then sketch a graph for the concentration of FSH over one period. If ovulation occurs around day 14, then what is the approximate concentration of FSH at that time?

b. Determine an equivalent FSH model of the form

$$F(t) = C + D\sin(\nu(t - \psi)),$$

where $D > 0$, $\nu > 0$, and $\psi \in [0, 28)$. Relate your constants in this model to the one above. Also, find a value of the phase shift $\psi \in [28, 56)$, which produces an equivalent model.

1.10 REFERENCES

[1] A. C. Guyton and J. E. Hall, *Textbook of Medical Physiology*, 9th edition, Philadelphia, W. B. Saunders (1996).

[2] J. M. Mahaffy and A. Chávez-Ross, *Calculus: A Mathematical Approach for the Life Sciences*, Volume I, San Diego State University, Pearson Custom Publishing (2009).