Calculus for the Life Sciences II

Lecture Notes – Trigonometric Functions

Joseph M. Mahaffy,
⟨mahaffy@math.sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://www-rohan.sdsu.edu/~jmahaffy

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Outline

Introduction — Trigonometric Functions

Introduction — Trigonometric Functions

- Many phenomena in biology appear in cycles
- Natural physical cycles
  - Daily cycle of light
  - Annual cycle of the seasons
- Oscillations are often modeled using trigonometric functions

Annual Temperature Cycles

Annual Temperature Cycles

- Weather reports give the average temperature for a day
- Long term averages help researchers predict effects of global warming over the background noise of annual variation
- There are seasonal differences in the average daily temperature
  - Higher averages in the summer
  - Lower averages in the winter

Modeling Annual Temperature Cycles

Modeling Annual Temperature Cycles

- What mathematical tools can help predict the annual temperature cycles?
- Polynomials and exponentials do not exhibit the periodic behavior
- Trigonometric functions exhibit periodicity

Average Temperatures for San Diego and Chicago 1

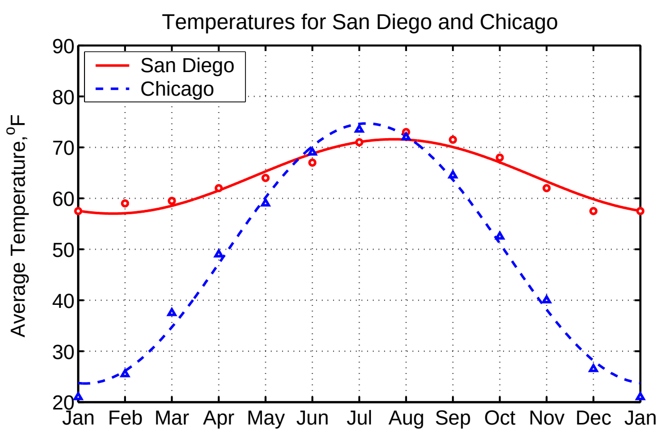
Average Temperatures for San Diego and Chicago:
Table of the monthly average high and low temperatures for San Diego and Chicago

Month	Jan	Feb	Mar	Apr	May	Jun
San Diego	66/49	67/51	66/53	68/56	69/59	72/62
Chicago	29/13	34/17	46/29	59/39	70/48	80/58

Month	Jul	Aug	Sep	Oct	Nov	Dec
San Diego	76/66	78/68	77/66	75/61	70/54	66/49
Chicago	84/63	82/62	75/54	63/42	48/32	34/19

Average Temperatures for San Diego and Chicago 2

Graph of Temperature for San Diego and Chicago with best fitting **trigonometric functions**



Average Temperatures for San Diego and Chicago 3

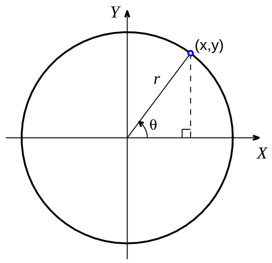
Models of Annual Temperature Cycles for San Diego and Chicago

- The two graphs have similarities and differences
  - Same seasonal period as expected
  - Seasonal variation or amplitude of oscillation for Chicago is much greater than San Diego
  - Overall average temperature for San Diego is greater than the average for Chicago
- Overlying models use **cosine functions**

Trigonometric Functions 1

Trigonometric Functions are often called **circular functions**

- Let (x, y) be a point on a circle of radius r centered at the origin
- Define the angle θ between the ray connecting the point to the origin and the x-axis



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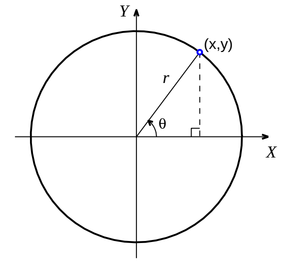
Trigonometric Functions

Trig Functions – **6 basic Trigonometric functions**



$$\sin(\theta) = \frac{y}{r} \qquad \cos(\theta) = \frac{x}{r} \qquad \tan(\theta) = \frac{y}{x}$$

$$\csc(\theta) = \frac{r}{y} \qquad \sec(\theta) = \frac{r}{x} \qquad \cot(\theta) = \frac{x}{y}$$

We will concentrate almost exclusively on the **sine** and **cosine**

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Radian Measure

Radian Measure

- Most trigonometry starts using degrees to measure an angle
  - This is **not** the appropriate unit to use in Calculus
- The **radian measure** of the angle uses the **unit circle**
- The distance around the perimeter of the unit circle is 2π
- The radian measure of the angle θ is simply the distance along the circumference of the unit circle
  - A 45° angle is $\frac{1}{8}$ the distance around the unit circle or $\frac{\pi}{4}$ radians
  - 90° and 180° angles convert to $\frac{\pi}{2}$ and π radians
- Conversions

$$1^\circ = \frac{\pi}{180} = 0.01745 \text{ radians} \quad \text{or} \quad 1 \text{ radian} = \frac{180^\circ}{\pi} = 57.296^\circ$$

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Sine and Cosine 1

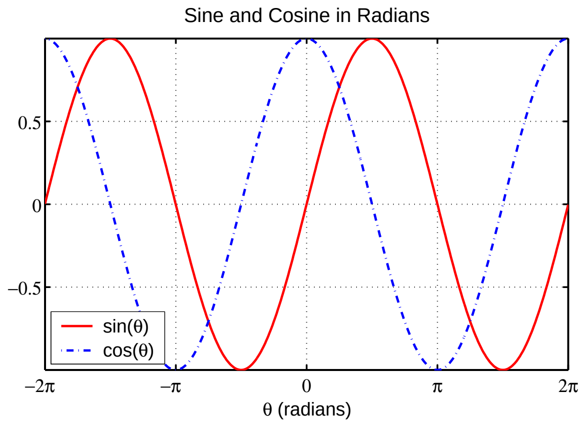
Sine and Cosine: The **unit circle** has $r = 1$, so the trig functions sine and cosine satisfy

$$\cos(\theta) = x \qquad \text{and} \qquad \sin(\theta) = y$$

- The formula for cosine (cos) gives the x value of the angle, θ, (measured in radians)
- The formula for sine (sin) gives the y value of the angle, θ
- The tangent function (tan) gives the slope of the line (y/x)

Sine and Cosine

Graph of $\sin(\theta)$ and $\cos(\theta)$ for angles $\theta \in [-2\pi, 2\pi]$



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Sine and Cosine

Sine and Cosine - Periodicity and Bounded

- Notice the 2π **periodicity** or the functions repeat the same pattern every 2π radians
  - This is clear from the circle because every time you go 2π radians around the circle, you return to the same point
- Note that both the sine and cosine functions are **bounded** between -1 and 1

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Sine and Cosine

Sine - Maximum and Minimum

- The **sine function** has its **maximum** value at $\frac{\pi}{2}$ with $\sin(\pi/2) = 1$
- By periodicity, $\sin(x) = 1$ for $x = \frac{\pi}{2} + 2n\pi$ for any integer n
- The **sine function** has its **minimum** value at $\frac{3\pi}{2}$ with $\sin(3\pi/2) = -1$
- By periodicity, $\sin(x) = -1$ for $x = \frac{3\pi}{2} + 2n\pi$ for any integer n

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Sine and Cosine

Cosine - Maximum and Minimum

- The **cosine function** has its **maximum** value at 0 with $\cos(0) = 1$
- By periodicity, $\cos(x) = 1$ for $x = 2n\pi$ for any integer n
- The **cosine function** has its **minimum** value at π with $\cos(\pi) = -1$
- By periodicity, $\cos(x) = -1$ for $x = (2n+1)\pi$ for any integer n

Sine and Cosine 6

Table of Some Important Values of Trig Functions

x	$\sin(x)$	$\cos(x)$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1

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Properties of Sine and Cosine 1

Properties of Cosine

- Periodic with **period** 2π
- Cosine is an **even** function
- Cosine is **bounded** by -1 and 1
- **Maximum** at $x = 0$, $\cos(0) = 1$
- By periodicity, other **maxima** at $x_n = 2n\pi$ with $\cos(2n\pi) = 1$ (n any integer)
- **Minimum** at $x = \pi$, $\cos(\pi) = -1$
- By periodicity, other **minima** at $x_n = (2n+1)\pi$ with $\cos(x_n) = -1$ (n any integer)
- **Zeroes of cosine** separated by π with $\cos(x_n) = 0$ when $x_n = \frac{\pi}{2} + n\pi$ (n any integer)

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Properties of Sine and Cosine 2

Properties of Sine

- Periodic with **period** 2π
- Sine is an **odd** function
- Sine is **bounded** by -1 and 1
- **Maximum** at $x = \frac{\pi}{2}$, $\sin\left(\frac{\pi}{2}\right) = 1$
- By periodicity, other **maxima** at $x_n = \frac{\pi}{2} + 2n\pi$ with $\sin(x_n) = 1$ (n any integer)
- **Minimum** at $x = \frac{3\pi}{2}$, $\sin\left(\frac{3\pi}{2}\right) = -1$
- By periodicity, other **minima** at $x_n = \frac{3\pi}{2} + 2n\pi$ with $\sin(x_n) = -1$ (n any integer)
- **Zeroes of sine** separated by π with $\sin(x_n) = 0$ when $x_n = n\pi$ (n any integer)

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Some Identities for Sine and Cosine

Some Identities for Cosine and Sine

- $\cos^2(x) + \sin^2(x) = 1$ for all values of x
- **Adding** and **Subtracting angles** for cosine

$$\begin{aligned} \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y) \end{aligned}$$

- **Adding** and **Subtracting angles** for sine

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y) \end{aligned}$$

Example of Shifts 1

Example of Shifts for Sine and Cosine:

Use the trigonometric identities to show

- $\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$
- $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$

The first example shows the cosine is the same as the sine function shifted to the left by $\frac{\pi}{2}$

The second example shows the sine is the same as the cosine function shifted to the right by $\frac{\pi}{2}$

Example of Shifts 2

Solution: We begin by using the additive identity for sine

$$\sin\left(x + \frac{\pi}{2}\right) = \sin(x)\cos\left(\frac{\pi}{2}\right) + \cos(x)\sin\left(\frac{\pi}{2}\right)$$

Since $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$,

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$

Example of Shifts 3

Solution (cont): Similarly from the additive identity for cosine

$$\cos\left(x - \frac{\pi}{2}\right) = \cos(x)\cos\left(\frac{\pi}{2}\right) + \sin(x)\sin\left(\frac{\pi}{2}\right)$$

Again $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$, so

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$

Trigonometric Models

Trigonometric Models are appropriate when data follows a simple oscillatory behavior

The Cosine Model

$$y(t) = A + B\cos(\omega(t - \phi))$$

The Sine Model

$$y(t) = A + B\sin(\omega(t - \phi))$$

Each model has **Four Parameters**

Vertical Shift and Amplitude

Trigonometric Model Parameters: For the cosine model

$$y(t) = A + B\,\cos(\omega(t-\phi))$$

- The model parameter A is the **vertical shift**, which is associated with the average height of the model
- The model parameter B gives the **amplitude**, which measures the distance from the average, A, to the maximum (or minimum) of the model

There are similar parameters for the sine model

Frequency and Period

Trigonometric Model Parameters: For the cosine model

$$y(t) = A + B\,\cos(\omega(t-\phi))$$

- The model parameter ω is the **frequency**, which gives the number of periods of the model that occur as t varies over 2π radians
- The **period** is given by $T = \dfrac{2\pi}{\omega}$

There are similar parameters for the sine model

Phase Shift

Trigonometric Model Parameters: For the cosine model

$$y(t) = A + B\,\cos(\omega(t-\phi))$$

- The model parameter ϕ is the **phase shift**, which shifts our models to the left or right
- This gives a **horizontal shift** of ϕ units
- If the period is denoted $T = \frac{2\pi}{\omega}$, then the **principle phase shift** satisfies $\phi \in [0, T)$
- By periodicity of the model, if ϕ is any **phase shift**

  $$\phi_1 = \phi + nT = \phi + \frac{2n\pi}{\omega}, \qquad n \text{ an integer}$$

  is a **phase shift** for an equivalent model

There is a similar parameter for the sine model

Model Parameters

Trigonometric Model Parameters: For the cosine and sine models

$$y(t) = A + B\,\cos(\omega(t-\phi))$$

and

$$y(t) = A + B\,\sin(\omega(t-\phi))$$

- The **vertical shift** parameter A is **unique**
- The **amplitude** parameter B is **unique** in magnitude but the sign can be chosen by the modeler
- The **frequency** parameter ω is **unique** in magnitude but the sign can be chosen by the modeler
- By periodicity, **phase shift** has infinitely many choices
- One often selects the **unique principle phase shift** satisfying $0 \le \phi < T$

Example: Period and Amplitude 1

Example 1: Consider the model

$$y(x) = 4\sin(2x)$$

Skip Example

- Find the period and amplitude
- Determine all maxima and minima for $x \in [-2\pi, 2\pi]$
- Sketch a graph

Example: Period and Amplitude 2

Solution: For

$$y(x) = 4\sin(2x)$$

- The **amplitude** is 4, so solution oscillates between -4 and 4
- The **frequency** is 2
- To find the period, let $x = T$
  - The argument of sine is $2x$, and the period of the sine function is 2π
  - The **period**, T, satisfies

$$2T = 2\pi \qquad \text{so} \qquad T = \pi$$

  - Alternately,

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$$

Example: Period and Amplitude 3

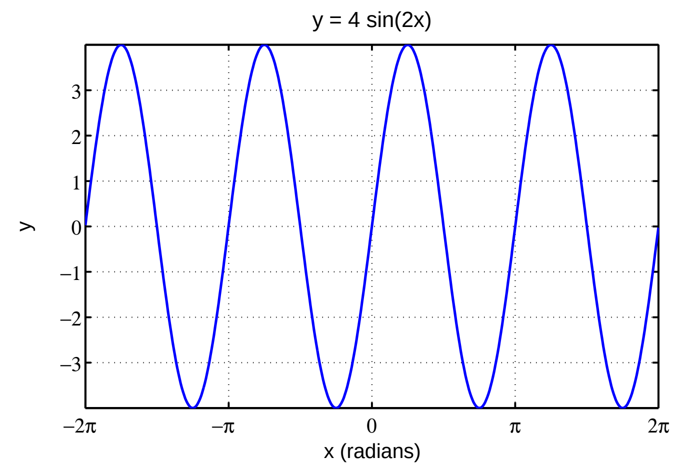
Solution (cont): For

$$y(x) = 4\sin(2x)$$

- The model begins at 0 when $x = 0$ and completes period at $x = \pi$
- Achieves a maximum of 4 when the argument $2x = \frac{\pi}{2}$ or $x = \frac{\pi}{4}$
- Achieves a minimum of -4 when the argument $2x = \frac{3\pi}{2}$ or $x = \frac{3\pi}{4}$
- By periodicity, other maxima at $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$, $x = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$, and $x = -\frac{9\pi}{4}$
- Similarly, there are other minima at $x = -\frac{5\pi}{4}, -\frac{\pi}{4}$, and $\frac{7\pi}{4}$.
- Sine is an **odd function**

Example: Period and Amplitude 4

Graph for



Example: Sine Function 1

Example 2: Consider the model

$$y(x) = 3\sin(2x) - 2$$

Skip Example

- Find the vertical shift, amplitude, and period
- Determine all maxima and minima for $x \in [-2\pi, 2\pi]$
- Sketch a graph

Example: Sine Function 2

Solution: For

$$y(x) = 3\sin(2x) - 2$$

- The **vertical shift** is -2
- The **amplitude** is 3, so solution oscillates between -5 and 1
- The **frequency** is 2
- The period, T, satisfies

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$$

Example: Sine Function 3

Solution (cont): For

$$y(x) = 3\sin(2x) - 2$$

- The model achieves a maximum of 1 when the argument $2x = \frac{\pi}{2}$ or $x = \frac{\pi}{4}$
- The model achieves a minimum of -5 when the argument $2x = \frac{3\pi}{2}$ or $x = \frac{3\pi}{4}$
- By periodicity, other maxima at $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$, $x = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$, and $x = -\frac{9\pi}{4}$
- Similarly, there are other minima at $x = -\frac{5\pi}{4}, -\frac{\pi}{4}$, and $\frac{7\pi}{4}$

Example: Sine Function 4

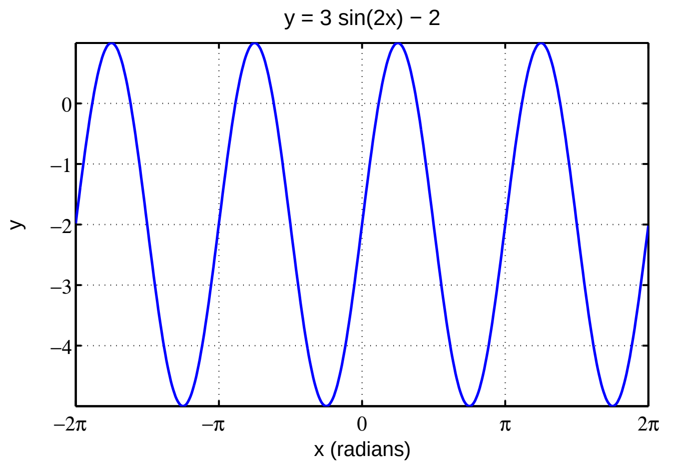
To **graph** a sine or cosine model, divide the period into 4 even parts

For this example, take $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

$$\begin{aligned}
y(0) &= 3\sin(2(0)) - 2 = 3\sin(0) - 2 = -2, \\
y(\pi/4) &= 3\sin(2(\pi/4)) - 2 = 3\sin(\pi/2) - 2 = 1, \\
y(\pi/2) &= 3\sin(2(\pi/2)) - 2 = 3\sin(\pi) - 2 = -2, \\
y(3\pi/4) &= 3\sin(2(3\pi/4)) - 2 = 3\sin(3\pi/2) - 2 = -5, \\
y(\pi) &= 3\sin(2(\pi)) - 2 = 3\sin(2\pi) - 2 = -2.
\end{aligned}$$

Example: Sine Function 5

Graph for



Example: Vertical Shift with Cosine Function 1

Example 3: Consider the model

$$y(x) = 3 - 2\cos(3x)$$

Skip Example

- Find the vertical shift, amplitude, and period
- Determine all maxima and minima for $x \in [0, 2\pi]$
- Sketch a graph

Example: Vertical Shift with Cosine Function 2

Solution: For

$$y(x) = 3 - 2\cos(3x)$$

- The **vertical shift** is 3
- The **amplitude** is 2 (noting that there is a negative sign), so solution oscillates between 1 and 5
- The **frequency** is 3
- The period, T, satisfies

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{3}$$

Example: Vertical Shift with Cosine Function 3

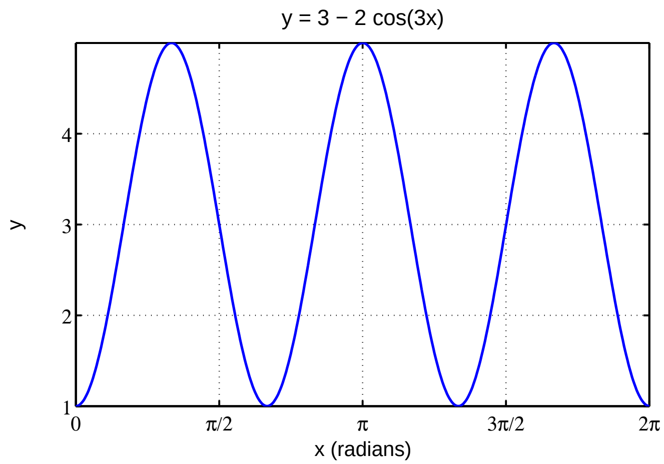
Solution (cont): For

$$y(x) = 3 - 2\cos(3x)$$

- The model achieves a minimum of 1 when the argument $3x = 0$ or $x = 0$
- The model achieves a maximum of 5 when the argument $3x = \pi$ or $x = \frac{\pi}{3}$
- By periodicity, the minima in the domain are $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, and 2π
- By periodicity, the maxima in the domain are $x = \frac{\pi}{3}, \pi$, and $\frac{5\pi}{3}$
- Note that this is an **even function**

Example: Vertical Shift with Cosine Function 4

Graph for



Example: Vertical Shift with Cosine Function 5

By inserting a **phase shift** of half a period, the constant for the **amplitude** becomes positive

$$y(x) = 3 + 2\cos\left(3(x - \tfrac{\pi}{3})\right).$$

Show this by employing the angle subtraction identity for the cosine function

$$\begin{aligned} y(x) &= 3 + 2\cos\left(3(x - \tfrac{\pi}{3})\right), \\ &= 3 + 2\cos(3x - \pi), \\ &= 3 + 2(\cos(3x)\cos(\pi) + \sin(3x)\sin(\pi)), \\ &= 3 - 2\cos(3x), \end{aligned}$$

since $\cos(\pi) = -1$ and $\sin(\pi) = 0$

Phase Shift in Models

Phase Shift of Half a Period

A **phase shift** of **half a period** creates an equivalent **sine** or **cosine model** with the sign of the **amplitude** reversed

Models Matching Data

- Phase shifts are important matching data in periodic models
- The **cosine model** is easiest to match, since the maximum of the cosine function occurs when the argument is zero

Example: Cosine Model with Phase Shift 1

Example 3: Consider the model

$$y(x) = 4 + 6\cos\left(\tfrac{1}{2}(x - \pi)\right), \qquad x \in [-4\pi, 4\pi]$$

Skip Example

- Find the vertical shift, amplitude, period, and phase shift
- Determine all maxima and minima for $x \in [0, 2\pi]$
- Sketch a graph
- Find the equivalent sine model

Example: Cosine Model with Phase Shift 2

Solution: Rewrite the model

$$y(x) = 4 + 6\cos\left(\tfrac{1}{2}(x - \pi)\right)$$

- The **vertical shift** is $A = 4$
- The **amplitude** is $B = 6$, so $y(x)$ oscillates between -2 and 10
- The **frequency** is $\omega = \frac{1}{2}$
- The **period**, T, satisfies

$$T = \frac{2\pi}{\omega} = 4\pi$$

- The **phase shift** is $\phi = \pi$, which means the cosine model is shifted horizontally $x = \pi$ units to the right
- Since cosine has a maximum with argument zero, a maximum will occur at $x = \pi$

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Example: Cosine Model with Phase Shift 3

Solution (cont): For graphing,

$$y(x) = 4 + 6\cos\left(\tfrac{1}{2}(x - \pi)\right)$$

The significant points are $x = \pi,\ 2\pi,\ 3\pi,\ 4\pi$, and 5π

$$\begin{aligned}
y(\pi) &= 4 + 6\cos\left(\tfrac{1}{2}(\pi - \pi)\right) = 4 + 6\cos(0) = 4 + 6(1) = 10,\\
y(2\pi) &= 4 + 6\cos\left(\tfrac{1}{2}(2\pi - \pi)\right) = 4 + 6\cos\left(\frac{\pi}{2}\right) = 4 + 6(0) = 4,\\
y(3\pi) &= 4 + 6\cos\left(\tfrac{1}{2}(2\pi - \pi)\right) = 4 + 6\cos(\pi) = 4 + 6(-1) = -2,\\
y(4\pi) &= 4 + 6\cos\left(\tfrac{1}{2}(2\pi - \pi)\right) = 4 + 6\cos\left(\frac{3\pi}{2}\right) = 4 + 6(0) = 4,\\
y(5\pi) &= 4 + 6\cos\left(\tfrac{1}{2}(2\pi - \pi)\right) = 4 + 6\cos(2\pi) = 4 + 6(1) = 10.
\end{aligned}$$

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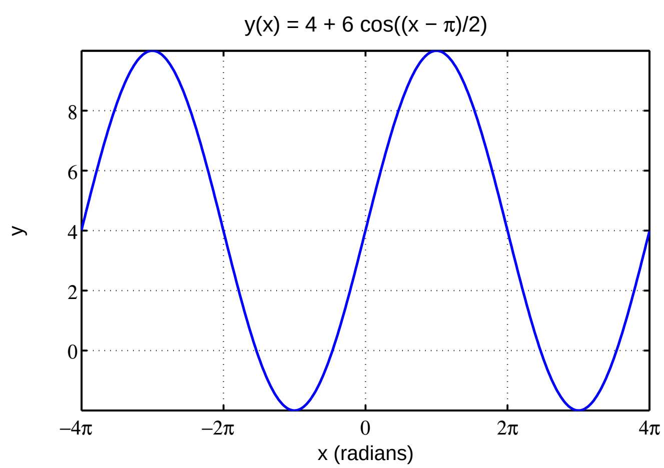
Example: Cosine Model with Phase Shift 4

Graph for



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Example: Cosine Model with Phase Shift 5

Solution (cont): The appropriate sine model has the same vertical shift, A, amplitude, B, and frequency, ω,

$$y(x) = 4 + 6\sin\left(\tfrac{1}{2}(x - \phi)\right)$$

Must find appropriate **phase shift**, ϕ

Recall the cosine function is horizontally shifted to the left of the sine function by $\frac{\pi}{2}$

$$\cos\left(\tfrac{1}{2}(x - \pi)\right) = \sin\left(\tfrac{1}{2}(x - \pi) + \tfrac{\pi}{2}\right) = \sin\left(\tfrac{1}{2}(x - \phi)\right)$$

It follows that we want

$$-\frac{\pi}{2} + \frac{\pi}{2} = -\frac{\phi}{2} \qquad \text{or} \qquad \phi = 0$$

Example: Cosine Model with Phase Shift 5

Solution (cont): The **equivalent sine model** is

$$y(x) = 4 + 6\sin\left(\frac{x}{2}\right)$$

Thus, the original **phase-shifted cosine model**

$$y(x) = 4 + 6\cos\left(\frac{1}{2}(x - \pi)\right)$$

is the same as an unshifted sine model

Equivalent Sine and Cosine Models

Phase Shift for Equivalent Sine and Cosine Models

Suppose that the sine and cosine models are equivalent, so

$$\sin(\omega(x - \phi_1)) = \cos(\omega(x - \phi_2)).$$

The relationship between the **phase shifts**, ϕ_1 and ϕ_2 satisfies:

$$\phi_1 = \phi_2 - \frac{\pi}{2\omega}.$$

Note: Remember that the phase shift is not unique
It can vary by integer multiples of the period, $T = \frac{2\pi}{\omega}$

Return to Annual Temperature Model 1

Annual Temperature Model: Started section with data and graphs of average monthly temperatures for Chicago and San Diego

- Fit data to cosine model for temperature, T,

$$T(m) = A + B\cos(\omega(m - \phi))$$

  where m is in months
- Find best model parameters, A, B, ω, and ϕ
- The frequency, ω, is constrained by a period of 12 months
- It follows that

$$12\omega = 2\pi \qquad \text{or} \qquad \omega = \frac{\pi}{6} = 0.5236$$

Return to Annual Temperature Model 2

Annual Temperature Model:

$$T(m) = A + B\cos(\omega(m - \phi))$$

- Choose A to be the average annual temperature
  - Average for San Diego is $A = 64.29$
  - Average for Chicago is $A = 49.17$
- Perform least squares best fit to data for B and ϕ
  - For San Diego, obtain $B = 7.29$ and $\phi = 6.74$
  - For Chicago, obtain $B = 25.51$ and $\phi = 6.15$

Return to Annual Temperature Model 3

Annual Temperature Model for San Diego:

$$T(m) = 64.29 + 7.29\cos(0.5236(m - 6.74))$$

Annual Temperature Model for Chicago:

$$T(m) = 49.17 + 25.51\cos(0.5236(m - 6.15))$$

- The **amplitude** of models
  - Temperature in San Diego only varies ±7.29°F, giving it a "Mediterranean" climate
  - Temperature in Chicago varies ±25.51°F, indicating cold winters and hot summers

Return to Annual Temperature Model 4

Annual Temperature Model for San Diego:

$$T(m) = 64.29 + 7.29\cos(0.5236(m - 6.74))$$

Annual Temperature Model for Chicago:

$$T(m) = 49.17 + 25.51\cos(0.5236(m - 6.15))$$

- The **phase shift** for the models
  - For San Diego, the phase shift of $\phi = 6.74$, so the maximum temperature occurs at 6.74 months (late July)
  - For Chicago, the phase shift of $\phi = 6.15$, so the maximum temperature occurs at 6.15 months (early July)

Return to Annual Temperature Model 5

Convert Cosine Model to Sine Model:

$$T(m) = A + B\sin(\omega(m - \phi_2))$$

- Formula shows
$$\phi_2 = \phi - \frac{\pi}{2\omega}$$
where ϕ is from the cosine model
  - For San Diego, $\phi_2 = 3.74$
  - For Chicago, $\phi_2 = 3.15$
- **Sine Model for San Diego:**
$$T(m) = 64.29 + 7.29\sin(0.5236(m - 3.74))$$
- **Sine Model for Chicago:**
$$T(m) = 49.17 + 25.51\sin(0.5236(m - 3.15))$$

Population Model with Phase Shift 1

Population Model: Suppose population data show a 10 year periodic behavior with a maximum population of 26 (thousand) at $t = 2$ and a minimum population of 14 (thousand) at $t = 7$

Assume a model of the form

$$y(t) = A + B\,\sin(\omega(t - \phi))$$

Skip Example

- Find the constants A, B, ω, and ϕ with $B > 0$, $\omega > 0$,and $\phi \in [0, 10)$
- Since ϕ is not unique, find values of ϕ with $\phi \in [-10, 0)$ and $\phi \in [10, 20)$
- Sketch a graph
- Find the equivalent cosine model

Population Model with Phase Shift 2

Solution: Compute the various parameters

- The **vertical shift** satisfies

$$A = \frac{26+14}{2} = 20$$

- The **amplitude** satisfies

$$B = 26 - 20 = 6$$

- Since the **period** is $T = 10$ years, the **frequency**, ω, satisfies

$$\omega = \frac{2\pi}{10} = \frac{\pi}{5}$$

Population Model with Phase Shift 3

Solution (cont): Compute the **phase shift**

- The maximum of 26 occurs at $t = 2$, so the model satisfies:

$$y(2) = 26 = 20 + 6\sin\left(\tfrac{\pi}{5}(2-\phi)\right)$$

- Clearly

$$\sin\left(\tfrac{\pi}{5}(2-\phi)\right) = 1$$

- The sine function is at its maximum when its argument is $\frac{\pi}{2}$, so

$$\begin{aligned} \tfrac{\pi}{5}(2-\phi) &= \tfrac{\pi}{2} \\ 2-\phi &= \tfrac{5}{2} \\ \phi &= -\tfrac{1}{2} \end{aligned}$$

Population Model with Phase Shift 4

Solution (cont): Continuing, the **phase shift** was

$$\phi = -\tfrac{1}{2}$$

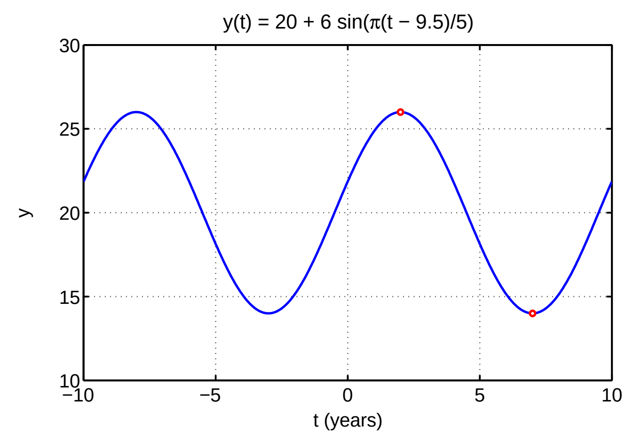
- This value of ϕ is not in the interval $[0, 10)$
- The periodicity, $T = 10$, of the model is also reflected in the phase shift, ϕ
-

$$\begin{aligned} \phi &= -\tfrac{1}{2} + 10\,n, \qquad n \text{ an integer} \\ \phi &= \dots -10.5, -0.5, 9.5, 19.5, \dots \end{aligned}$$

- The **principle phase shift** is $\phi = 9.5$

Population Model with Phase Shift 5

Solution (cont): The **sine model** is



Population Model with Phase Shift 6

Solution (cont): The **cosine model** has the form

$$y(t) = 20 + 6\cos\left(\frac{\pi}{5}(t - \phi_2)\right),$$

- The vertical shift, amplitude, and frequency match the sine model
- The maximum of the cosine function occurs when its argument is zero, so

$$\begin{aligned} \frac{\pi}{5}(2 - \phi_2) &= 0, \\ \phi_2 &= 2. \end{aligned}$$

- The cosine model satisfies

$$y(t) = 20 + 6\cos\left(\frac{\pi}{5}(t - 2)\right)$$

Body Temperature 1

Circadian Rhythms:

- Humans, like many organisms, undergo **circadian rhythms** for many of their bodily functions
- Circadian rhythms are the daily fluctuations that are driven by the light/dark cycle of the Earth
- Seems to affect the pineal gland in the head
- This temperature normally varies a few tenths of a degree in each individual with distinct regularity
- The body is usually at its hottest around 10 or 11 AM and at its coolest in the late evening, which helps encourage sleep

Body Temperature 2

Body Temperature Model: Suppose that measurements on a particular individual show

- A high body temperature of 37.1°C at 10 am
- A low body temperature of 36.7°C at 10 pm

Assume body temperature T and a model of the form

$$T(t) = A + B\cos(\omega(t - \phi))$$

- Find the constants A, B, ω, and ϕ with $B > 0$, $\omega > 0$,and $\phi \in [0, 24)$
- Graph the model
- Find the equivalent sine model

Body Temperature 3

Solution: Compute the various parameters

- The **vertical shift** satisfies

$$A = \frac{37.1 + 36.7}{2} = 36.9$$

- The **amplitude** satisfies

$$B = 37.1 - 36.9 = 0.2$$

- Since the **period** is $P = 24$ hours, the **frequency**, ω, satisfies

$$\omega = \frac{2\pi}{24} = \frac{\pi}{12}$$

Body Temperature 4

Solution (cont): Compute the **phase shift**

- The maximum of 37.1°C occur at $t = 10$ am
- The cosine function has its maximum when its argument is 0 (or any integer multiple of 2π)
- The appropriate phase shift solves

$$\omega(10 - \phi) = 0 \qquad \text{or} \qquad \phi = 10$$

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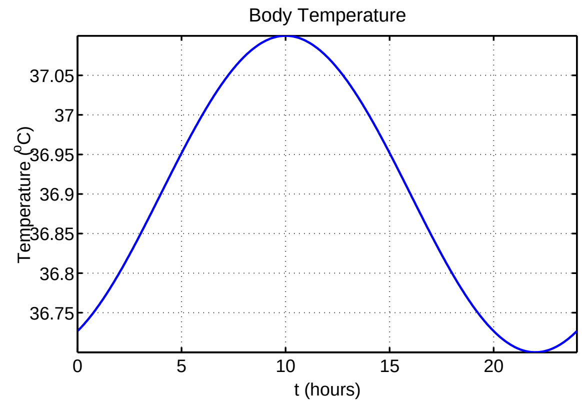
Body Temperature 5

Solution (cont): The **cosine model** is

$$T(t) = 36.9 + 0.2\cos\left(\frac{\pi}{12}(t - 10)\right)$$



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Body Temperature 6

Solution (cont): The **sine model** for body temperature is

$$T(t) = 36.9 + 0.2\sin\left(\frac{\pi}{12}(t - \phi_2)\right)$$

- The vertical shift, amplitude, and frequency match the cosine model
- From our formula above

$$\phi_2 = 10 - \frac{\pi}{2\omega} = 10 - 6 = 4$$

- The sine model satisfies

$$T(t) = 36.9 + 0.2\sin\left(\frac{\pi}{12}(t - 4)\right)$$