Calculus for the Life Sciences II

Lecture Notes – Separable Diferential Equations

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Outline

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  - Malthusian Growth and Radioactive decay - Solution recognition

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  - Numerical methods - Handles ones not solvable by other means

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  - Malthusian Growth and Radioactive decay - Solution recognition
  - Newton's Law of Cooling and Mixing Problems - Substitution to create above form
  - Time varying - Solved by integration
  - Numerical methods - Handles ones not solvable by other means
- This section examines a class of differential equations that separate into two integration problems for their solution

Malthusian Growth Model

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- A simple Malthusian growth model with a single growth rate is very limited in applications
- For discrete Malthusian growth models, a time varying component added to the model predicts the population much more accurately
- Time varying growth rates are very appropriate for human populations accounting for changes in growth rates due to changing societal conditions

Malthusian Growth Model for U. S.

Malthusian Growth Model for U. S. Consider constant growth model

$$\frac{dP(t)}{dt} = r\,P(t), \quad \text{with} \quad P(t_0) = P_0$$

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- Since $P(200) = 248.7$, then $r = \left(\frac{1}{200}\right)\ln\left(\frac{248.7}{3.93}\right) = 0.02074$, so

$$P(t) = 3.93\,e^{0.02074t}$$

Malthusian Growth Model for U. S. 2

Modify Malthusian Growth Model Consider time-varying growth model

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- This is still a **linear differential equation**
- How do we solve this type of differential equation?
- What are the best constants a and b that fit the data for the U. S. population in 1790 and 1990?
- We must first learn about **Separable Differential Equations**

Separable Differential Equations 1

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$$f(t, y) = M(t)N(y)$$

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- Assume the function $f(t,y)$ has the special separable form with
$$f(t,y) = M(t)N(y)$$
- Think of $\frac{dy}{dt}$ as the quotient of differentials
- We separate the differential equation in the following manner:

$$\begin{aligned} \frac{dy}{dt} &= M(t)N(y) \\ \frac{dy}{N(y)} &= M(t)dt \end{aligned}$$

Separable Differential Equations 2

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Separable Differential Equations The differential equation

$$\frac{dy}{dt} = M(t)N(y)$$

- Separate so the left hand side has only the **dependent variable**, y, and the right hand side has only the **independent variable**, t
- The solution is obtained by integrating both sides

$$\int \frac{dy}{N(y)} = \int M(t)dt$$

Example 1 - Separable Differential Equation 1

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$$\frac{dy}{dt} = 2ty^2$$

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- Separate the variables t and y
  - Put only $2t$ and dt on the right hand side
  - And only y^2 and dy are on the left hand side
- The integral form is

$$\int \frac{dy}{y^2} = \int 2t\,dt$$

Example 1 - Separable Differential Equation 2

Solution (cont) The two integrals are

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- The two integrals are easily solved

$$-\frac{1}{y} = t^2 + C$$

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- **Note** that you only need to put **one arbitrary constant**, despite solving two integrals
- This is easily rearranged to give the solution in explicit form

$$y(t) = -\frac{1}{t^2 + C}$$

Example 2 - Separable Differential Equation 1

Example 2: Consider the initial value problem

$$\frac{dy}{dt} = \frac{4\,\sin(2t)}{y} \quad \text{with} \quad y(0) = 1$$

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$$\frac{y^2}{2} = -2\cos(2t) + C$$

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We write

$$y^2(t) = 2C - 4\cos(2t) \qquad \text{or} \qquad y(t) = \pm\sqrt{2C - 4\cos(2t)}$$

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From the initial condition

$$y(0) = 1 = \sqrt{2C - 4\cos(0)} = \sqrt{2C - 4}$$

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Thus, $2C = 5$, and

$$y(t) = \sqrt{5 - 4\cos(2t)}$$

Example 3 - Separable Differential Equation 1

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$$\frac{dy}{dt} = -y\frac{(1+2\,t^2)}{t} \quad \text{with} \quad y(1) = 2$$

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Exponentiate both sides to give

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Exponentiate both sides to give

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$$y(t) = e^{-\ln(t)-t^2+C} = e^{-\ln(t)}e^{-t^2}e^C = \frac{A}{t}e^{-t^2}$$

where $A = e^C$

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$$y(1) = 2 = A\, e^{-1} \quad \text{or} \quad A = 2\, e^1$$

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The solution is

$$y(t) = \frac{2}{t}e^{1-t^2}$$

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- Integrating

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- Exponentiating

$$P(t) = e^{\left(\frac{a\,t^2}{2} + b\,t + C\right)}$$

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- As before, we fit the census data in 1790 aand 1990 of 3.93 million and 248.7 million
- Choose the third data value from the census in 1890, where the population is 62.95 million
- Again take t to be the years after 1790, then $P_0 = 3.93$

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Population Model for U. S. The nonautonomous model is

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$$P(t) = 3.93\, e^{\left(\frac{a\,t^2}{2} + b\,t\right)}$$

- Use the census data in 1890 and 1990 to find a and b
- The model gives

$$\begin{aligned} P(100) &= 62.95 = 3.93\, e^{5000a + 100b} \\ P(200) &= 248.7 = 3.93\, e^{20000a + 200b} \end{aligned}$$

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$$\begin{aligned} P(100) &= 62.95 = 3.93\, e^{5000a+100b} \\ P(200) &= 248.7 = 3.93\, e^{20000a+200b} \end{aligned}$$

- Taking logarithms, we have the linear equations

$$\begin{aligned} 5000\,a + 100\,b &= \ln\left(\frac{62.95}{3.93}\right) = 2.7737 \\ 20,000\,a + 200\,b &= \ln\left(\frac{248.7}{3.93}\right) = 4.1476 \end{aligned}$$

Modified Malthusian Growth Model 3

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- Multiply the first equation by -2 and add to the second

$$10,000\,a = -2(2.7737) + 4.1476 = -1.3998$$

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- It follows that

$$100\,b = 5000(0.00013998) + 2.7737 = 3.473$$

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- It follows that

$$100\,b = 5000(0.00013998) + 2.7737 = 3.473$$

- Solution is $a = -0.00013998$ and $b = 0.03473$

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The nonautonomous model fitting the census data at 1790, 1890, and 1990 is

$$P(t) = 3.93\, e^{0.03474t - 0.00006999t^2}$$

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Model	1900	2000	2010
U. S. Census Data	76.21	281.4	308.7
Malthusian Growth	38.48	306.1	376.7
Nonautonomous	76.95	264.4	277.0

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- For 2000 and 2010
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  - The **nonautonomous growth model** is too low by 6.0% and 10.3%

Modified Malthusian Growth Model 5

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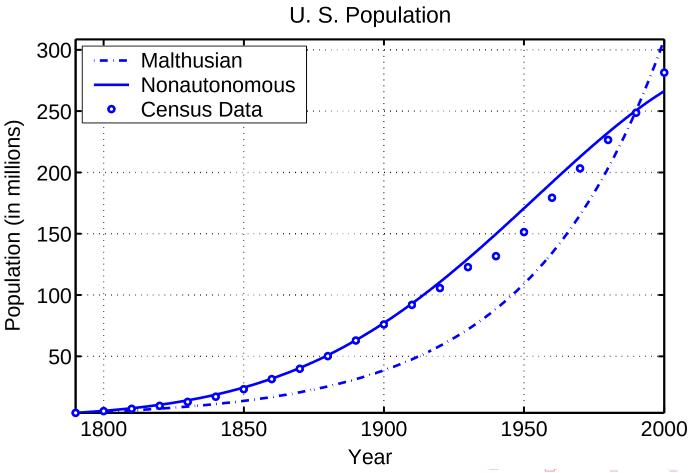
Modified Malthusian Growth Model

Graphs of Population Models for U. S.



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Population of Italy (cont):

- If the population of Italy was 50.2 million in 1960 and 57.6 million in 2000, then use each of these models to estimate the populations and determine the error between the models and the actual census values

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Solution: The Malthusian growth model satisfies

$$\frac{dP}{dt} = rP \qquad \text{with} \qquad P(0) = 47.1$$

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Solution (cont): The solution of the Malthusian growth model is

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- The Malthusian growth model for Italy is

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Population of Italy

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$$\frac{at^2}{2} + bt = \ln(P(t)) - \ln(47.1)$$

- The data from 1970 and 1990 give

$$\begin{aligned} 200\,a + 20\,b &= \ln(53.7) - \ln(47.1) = 0.13114 \\ 800\,a + 40\,b &= \ln(56.8) - \ln(47.1) = 0.18726 \end{aligned}$$

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- The solution becomes

$$P(t) = 47.1\,e^{0.0084325\,t - 0.00009378\,t^2}$$

Population of Italy

Solution (cont): The two models are given by

$$P(t) = 47.1\, e^{0.004682\, t} \quad \text{and} \quad P(t) = 47.1\, e^{0.0084325\, t - 0.00009378\, t^2}$$

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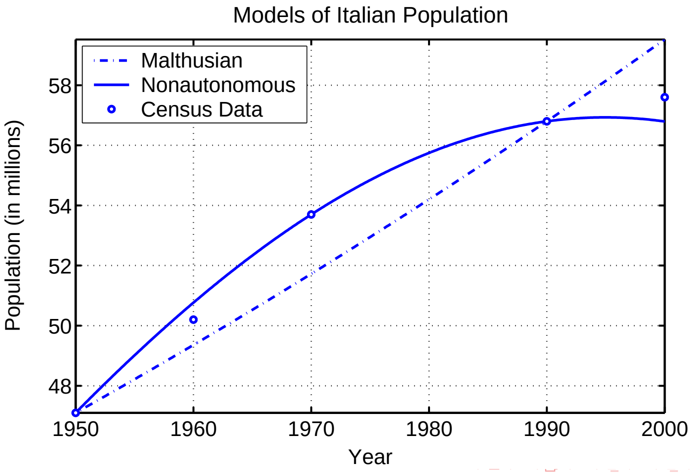
Below is a Table comparing the models at 1960 and 2000

Model	1960	% Error	2000	% Error
Italy Census Data	50.2	–	57.6	–
Malthusian	49.4	−1.7%	59.5	3.3%
Nonautonomous	50.8	1.1%	56.8	−1.4%

Modified Malthusian Growth Model

Graphs of Population Models for Italy



Population of Italy

Solution (cont): The nonautonomous model is

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Population of Italy 9

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- An appropriate model for the dessication of a cell is

$$\frac{dV}{dt} = -kV^{2/3}$$

where $V(t)$ is the volume of the cell

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Solution: The model is a separable differential equation

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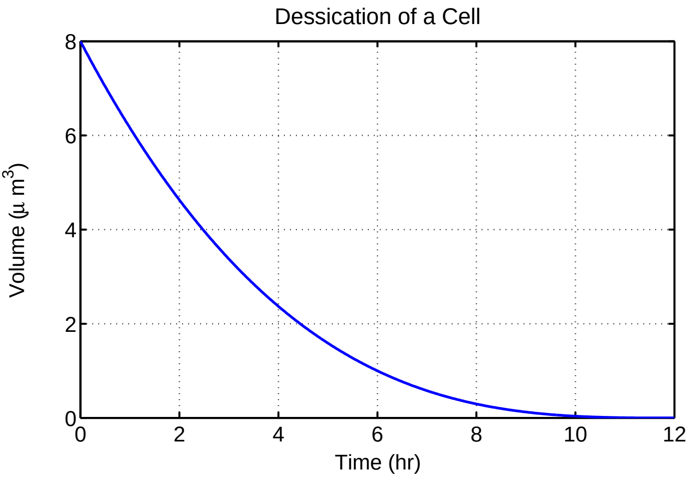
- The solution vanishes (all the water evaporates) at $t = 12$

Dessication of a Cell

Graphs of Dessication of a Cell



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- Water flowing from a hole in the bottom of a reservoir of water satisfies Torricelli's law

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- This equation is derived using basic physics with the assumption that the sum of the kinetic and potential energy of the system remains constant

Water Height - Torricelli's Law

Modeling Water Height: The volume of water in the reservoir is equal to the cross-sectional area (A) of the cylinder times the height of the water (h) with A constant and $h(t)$ varying with time

$$V(t) = A\,h(t)$$

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- It follows that
$$\frac{dV}{dt} = A\frac{dh}{dt}$$
- By Torricelli's Law, the model for the height is
$$\frac{dh}{dt} = -\frac{k}{A}\sqrt{h}$$

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Modeling Water Height: Suppose that a reservoir with a 20 cm radius begins with a height of 144 cm of water satisfies Torricelli's Law

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- Determine how long until the reservoir is empty

Water Height - Torricelli's Law (4)

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Solution: The model is

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- This equation is solved explicitly for $h(t)$

$$h(t) = \left(\frac{C}{2} - 0.0125\,t\right)^2$$

Water Height - Torricelli's Law

Solution (cont): The model is

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Water Height - Torricelli's Law (6)

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$$h(t) = \left(\frac{C}{2} - 0.0125\,t\right)^2$$

- From the initial condition,

$$h(0) = 144 = \left(\frac{C}{2}\right)^2 \quad \text{or} \quad C = 24$$

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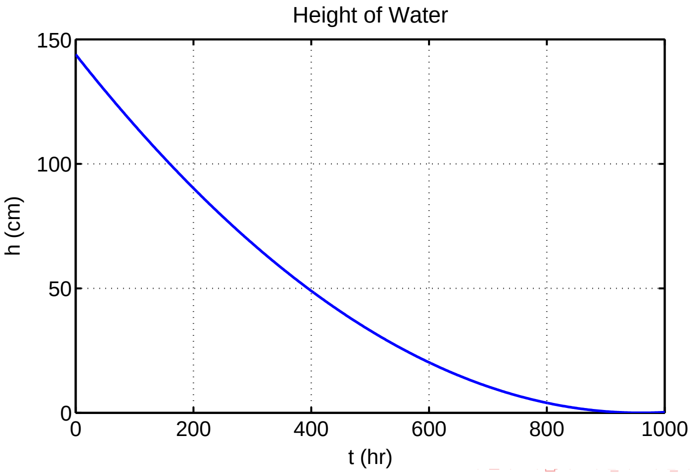
- The solution is

$$h(t) = (12 - 0.0125\,t)^2$$

Water Height - Torricelli's Law

Graphs of Water Height



Water Height - Torricelli's Law

Solution (cont): The reservoir is empty when

$$h(t) = (12 - 0.0125\,t)^2 = 0$$

Water Height - Torricelli's Law

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$$V = \pi(20)^2 144 = 57,600\pi = 180,956 \text{ cm}^3$$

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- The total volume in the reservoir is

$$V = \pi(20)^2 144 = 57,600\pi = 180,956 \text{ cm}^3$$

- The average amount of water delivered over 960 hr is $\frac{180{,}956}{960} = 188.5 \text{ cm}^3/\text{hr}$

Water Height - Torricelli's Law

Solution (cont): The differential equation is used to find the water delivered at 100 and 800 hr

$$\frac{dV}{dt} = -0.025\, A\sqrt{h}$$

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Water Height - Torricelli's Law

Solution (cont): The differential equation is used to find the water delivered at 100 and 800 hr

$$\frac{dV}{dt} = -0.025\, A\sqrt{h}$$

- The cross-sectional area satisfies

$$A = \pi(20)^2 = 400\pi = 1257 \text{ cm}^2$$

- The height of the water at $t = 100$ and 800 hr is

$$\begin{aligned} h(100) &= (12 - 0.0125(100))^2 = 115.6 \text{ cm} \\ h(800) &= (12 - 0.0125(800))^2 = 4.0 \text{ cm} \end{aligned}$$

Water Height - Torricelli's Law

Solution (cont): The volume of water flowing out is

$$\frac{dV}{dt} = -0.025(1257)\sqrt{h}$$

Water Height - Torricelli's Law

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$$\frac{dV}{dt} = -0.025(1257)\sqrt{115.6} = -337.7 \text{ cm}^3/\text{hr}$$

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- The volume flowing out of the reservoir at $t = 800$ satisfies

$$\frac{dV}{dt} = -0.025(1257)\sqrt{4.0} = -62.85 \text{ cm}^3/\text{hr}$$

Water Height - Torricelli's Law

Solution (cont): The volume of water flowing out is

$$\frac{dV}{dt} = -0.025(1257)\sqrt{h}$$

- The volume flowing out of the reservoir at $t = 100$ satisfies

$$\frac{dV}{dt} = -0.025(1257)\sqrt{115.6} = -337.7 \text{ cm}^3/\text{hr}$$

- This is above the average rate of water flowing out
- The volume flowing out of the reservoir at $t = 800$ satisfies

$$\frac{dV}{dt} = -0.025(1257)\sqrt{4.0} = -62.85 \text{ cm}^3/\text{hr}$$

- This is below the average rate of water flowing out