Calculus for the Life Sciences II

Lecture Notes – Riemann Sums and Numerical Integration

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Outline

Introduction

Introduction

- We need a proper definition for the integral
- Riemann sums provide the basis for the integral
- The integral represents the area under a curve
- The proper definition suggests means to numerically compute the integral

Salton Sea 1

Salton Sea: One of the world's largest inland seas created by accident when a dike broke during the construction of the All-American Canal in 1905

- Popular recreation area for boating and fishing
- Crucial region for birds on migration because loss of water habitat
- Sea is 228 ft below sea level, so water only lost by evaporation
- Agricultural activities result in serious pollution problems

Salton Sea 2

Area of Salton Sea: How can we determine the area of the Salton sea?

- One technique is to cut out the image of the lake and weigh it against a standard measured area
- Computers have advanced software that measure the area quite accurately by a simple scanning or tracing process
- Place a refined grid on the picture and determine the area
- All these schemes use the process of **integration**

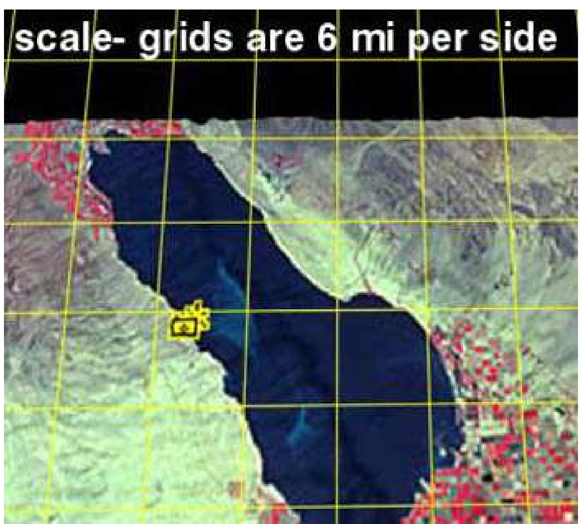
Salton Sea 3

Area of Salton Sea: Use a gridding scheme over an image

- The area is determined by counting the number of squares that include the image of the Salton Sea
  - If a box is at least 50% full, we will count it
  - If a box is less than 50% full, we will not count it
- As the boxes get smaller the estimate of the area of the Salton Sea becomes more accurate

Salton Sea 4

Salton Sea grid with 6 mi on a side



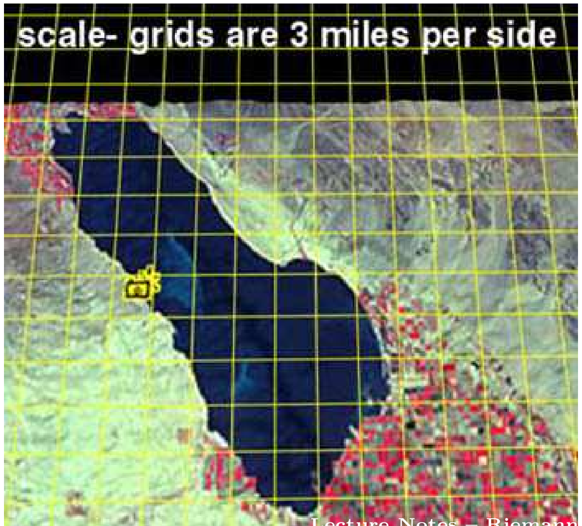
Salton Sea 5

Area of Salton Sea: Using the 6 mi square grid with 50% rule

- 8 squares apply to this rule
- Each square is a 36 square mile area
- This approximation gives 288 square miles
- Assuming the actual area of the basin is 360 square miles, the error is 20%

Salton Sea

Salton Sea grid with 3 mi on a side



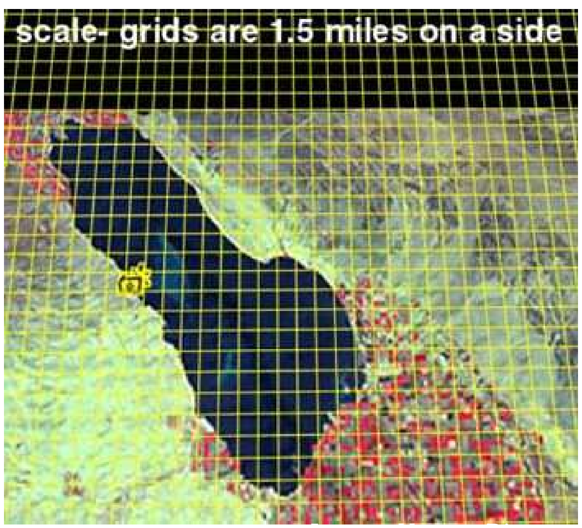
Salton Sea

Area of Salton Sea: Using the 3 mi square grid with 50% rule

- 33 squares apply to this rule
- Each square is a 9 square mile area
- This approximation gives 297 square miles
- Assuming the actual area of the basin is 360 square miles, the error is 17.5%

Salton Sea

Salton Sea grid with 1.5 mi on a side



Salton Sea

Area of Salton Sea: Using the 1.5 mi square grid with 50% rule

- 137 squares apply to this rule
- Each square is a 2.25 square mile area
- This approximation gives 308.25 square miles
- Assuming the actual area of the basin is 360 square miles, the error is 14%
- From the figure it is easy to see that shrinking the squares gives a better and better approximation of the area

Area under a Curve 1

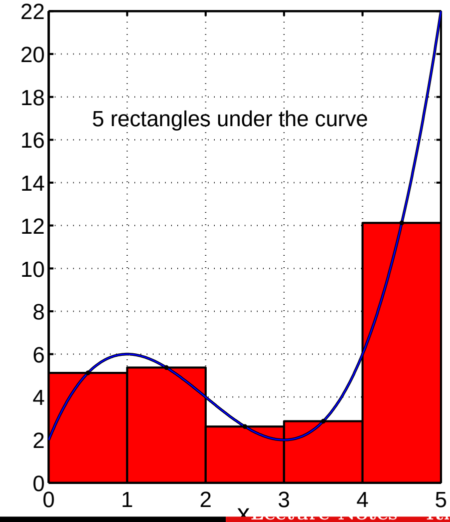
Area under a Curve: Consider the function

$$f(x) = x^3 - 6\,x^2 + 9\,x + 2 \qquad \text{for} \quad x \in [0,5]$$

- The actual area under the curve is **28.75**
- Approximate area with rectangles under the curve
- Divide the interval $x \in [0,5]$ into even intervals
- Use the midpoint of the interval to get height of the rectangle
- Examine approximation as intervals get smaller

Area under a Curve 2

Area under a Curve Divide $x \in [0,5]$ into 5 intervals



Area under a Curve 3

Area under a Curve: Height of rectangles from the function

$$f(x) = x^3 - 6\,x^2 + 9\,x + 2 \qquad \text{for} \quad x \in [0,5]$$

- Width of the rectangles are $\Delta x = 1$
- Height of rectangles evaluated at midpoints
- Approximate area satisfies

$$A \approx \left(f\left(\tfrac{1}{2}\right) + f\left(\tfrac{3}{2}\right) + f\left(\tfrac{5}{2}\right) + f\left(\tfrac{7}{2}\right) + f\left(\tfrac{9}{2}\right)\right)\Delta x = \sum_{i=0}^{4} f\left(i + \tfrac{1}{2}\right) \cdot 1$$

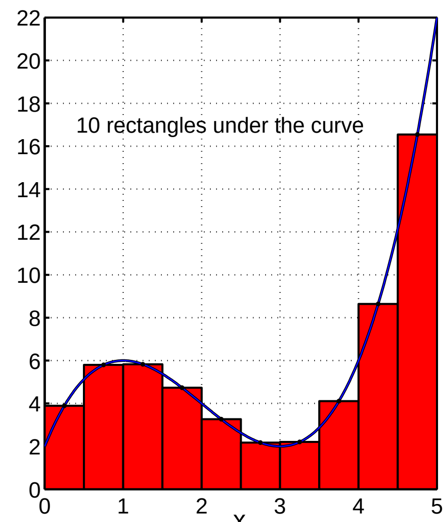
- This gives

$$A \approx \sum_{i=0}^{4}\left(\left(i+\tfrac{1}{2}\right)^3 - 6\left(i+\tfrac{1}{2}\right)^2 + 9\left(i+\tfrac{1}{2}\right) + 2\right) = 28.125$$

- This is 2.17% less than the actual area

Area under a Curve 4

Area under a Curve Divide $x \in [0,5]$ into 10 intervals



Area under a Curve

Area under a Curve: Height of rectangles from the function

$$f(x) = x^3 - 6\,x^2 + 9\,x + 2 \qquad \text{for} \quad x \in [0,5]$$

- Width of the rectangles are $\Delta x = \frac{1}{2}$
- Height of rectangles evaluated at midpoints
- Approximate area satisfies

$$A \approx \sum_{i=0}^{9} f\left(\tfrac{i}{2} + \tfrac{1}{4}\right)\Delta x$$

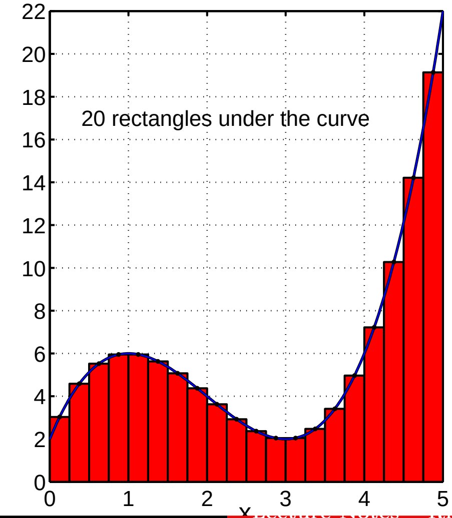
- This gives

$$A \approx \frac{1}{2}\sum_{i=0}^{9}\left(\left(\tfrac{i}{2}+\tfrac{1}{4}\right)^3 - 6\left(\tfrac{i}{2}+\tfrac{1}{4}\right)^2 + 9\left(\tfrac{i}{2}+\tfrac{1}{4}\right) + 2\right) = 28.59375$$

- This is 0.543% less than the actual area

Area under a Curve

Area under a Curve Divide $x \in [0,5]$ into 20 intervals



Area under a Curve

Area under a Curve: Height of rectangles from the function

$$f(x) = x^3 - 6\,x^2 + 9\,x + 2 \qquad \text{for} \quad x \in [0,5]$$

- Width of the rectangles are $\Delta x = \frac{1}{4}$
- Height of rectangles evaluated at midpoints
- Approximate area satisfies

$$A \approx \sum_{i=0}^{19} f\left(\tfrac{i}{4} + \tfrac{1}{8}\right)\Delta x$$

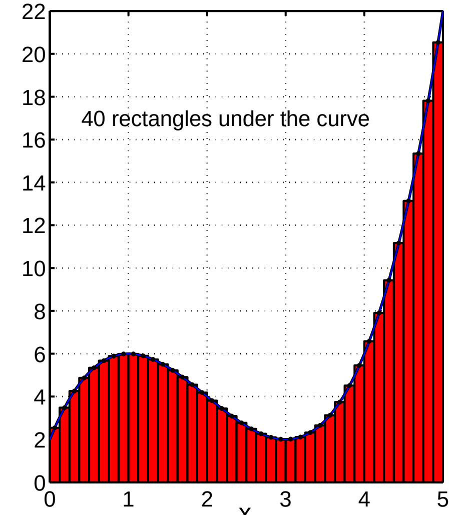
- This gives

$$A \approx \frac{1}{4}\sum_{i=0}^{19}\left(\left(\tfrac{i}{4}+\tfrac{1}{8}\right)^3 - 6\left(\tfrac{i}{4}+\tfrac{1}{8}\right)^2 + 9\left(\tfrac{i}{4}+\tfrac{1}{8}\right) + 2\right) = 28.7109$$

- This is 0.135% less than the actual area

Area under a Curve

Area under a Curve Divide $x \in [0,5]$ into 40 intervals



Area under a Curve 9

Area under a Curve: Height of rectangles from the function

$$f(x) = x^3 - 6\,x^2 + 9\,x + 2 \qquad \text{for} \quad x \in [0, 5]$$

- Width of the rectangles are $\Delta x = \frac{1}{8}$
- Height of rectangles evaluated at midpoints
- Approximate area satisfies

$$A \approx \sum_{i=0}^{39} f\left(\tfrac{i}{8} + \tfrac{1}{16}\right)\Delta x$$

- This gives

$$A \approx \frac{1}{8}\sum_{i=0}^{39}\left(\left(\tfrac{i}{8} + \tfrac{1}{16}\right)^3 - 6\left(\tfrac{i}{8} + \tfrac{1}{16}\right)^2 + 9\left(\tfrac{i}{8} + \tfrac{1}{16}\right) + 2\right) = 28.7402$$

- This is 0.034% less than the actual area

Definition of Riemann Integral 1

Definition of Riemann Integral: Suppose that we want to find the area under some continuous function $f(x)$ between $x = a$ and $x = b$

- Divide the interval $[a, b]$ into a large number of very small intervals
- For simplicity of discussion, divide the interval into n even intervals (though Riemann sums do not require this restriction)
- Also, for simplicity, evaluate the function, $f(x)$, at the midpoint of any subinterval
- Technically, it is important that one could arbitrarily take any point in the interval, but that is beyond the scope of this course

Definition of Riemann Integral 2

- Let $x_0 = a$ and $x_n = b$ and define $\Delta x = \frac{b-a}{n}$ with $x_i = a + i\Delta x$ for $i = 0, ..., n$
- This **partitions the interval** $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ each with length Δx
- The midpoint of each of these intervals is given by

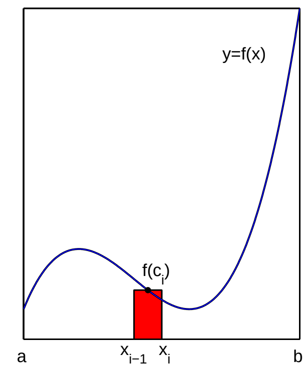
$$c_i = \frac{x_i + x_{i-1}}{2}$$

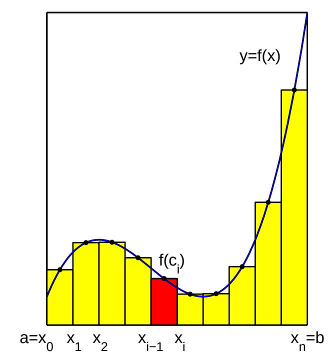
- The height of the approximating rectangle is found by evaluating the function at the midpoint, c_i
- The area of the rectangle, R_i, over the interval $[x_{i-1}, x_i]$ is given by its height times its width or

$$R_i = f(c_i)\Delta x$$

Definition of Riemann Integral 3

Figures below show a single rectangle in computing area of the **Riemann Integral** and all of the rectangles using the **Midpoint Rule** for approximating the area under the curve





Definition of Riemann Integral 4

Midpoint Rule for Integration is a method for approximating integrals

- Consider a continuous function $f(x)$ and an interval $x \in [a, b]$
- Subdivide the interval into n pieces, evaluating the function at the midpoints
- The area under $f(x)$ is approximated by adding the areas of the rectangles

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta x$$

- This is the **Midpoint Rule for Integration**
- Like **Euler's Method**, there are much better numerical methods for integration

Definition of Riemann Integral 5

Riemann Sums and Riemann Integral

- The **Midpoint Rule** described above is a specialized form of **Riemann sums**
- The more general form of Riemann sums allows the subintervals to have varying lengths, Δx_i
- The choice of where the function is evaluated need not be at the midpoint as described above
- The **Riemann integral** is defined using a limiting process, similar to the one described above

Definition of Riemann Integral 6

- Let $f(x)$ be a continuous function in the interval $[a, b]$
- Partition the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ with $\Delta x_i = x_i - x_{i-1}$ and Δx_k being the largest
- Let c_i be some point in the subinterval $[x_{i-1}, x_i]$
- The n^{th} **Riemann sum** is given by

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

- The **Riemann integral** is defined by

$$\int_a^b f(x)dx = \lim_{\Delta x_k \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Numerical Methods for Integration

Numerical Methods for Integration

- Many integrals cannot be solved exactly
- The Riemann integral has a number of methods for finding approximate solutions
- The Riemann integral represents the area under a function on a specified interval
- This is a **definite integral**

$$\int_a^b f(x)dx$$

Midpoint Rule

Midpoint Rule was discussed above and is reviewed below

- Let $f(x)$ be a continuous function on the interval $[a,b]$
- The interval of integration $[a,b]$ is divided into n subintervals $[x_{i-1}, x_i]$ with length $\Delta x = \frac{b-a}{n}$
- The midpoint of each of these intervals is $c_i = \frac{x_i + x_{i-1}}{2}$
- Height of an approximating rectangle, $f(c_i)$
- The **Midpoint Rule** satisfies

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(c_i)\Delta x$$

Trapezoid Rule 1

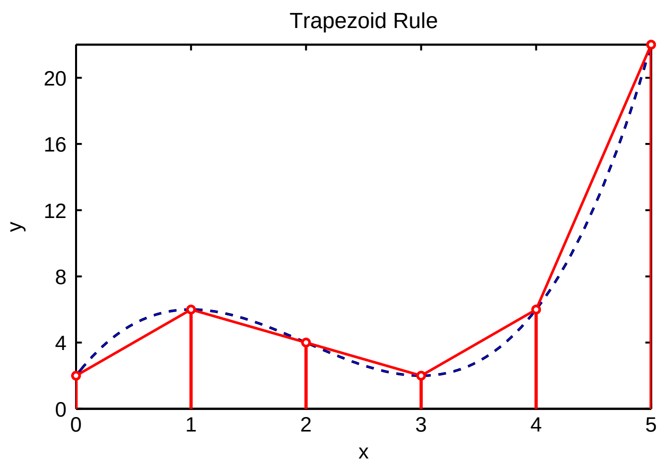
Trapezoid Rule approximates the area under a curve using trapezoids

- Let $f(x)$ be a continuous function on the interval $[a,b]$
- The interval of integration $[a,b]$ is divided into n subintervals $[x_{i-1}, x_i]$ with length $\Delta x = \frac{b-a}{n}$
- The function is evaluated at the endpoints of the subintervals
- A line segment is formed between these function evaluations on each subinterval creating a trapezoid
- The **Trapezoid Rule** satisfies

$$\int_a^b f(x)dx \approx \left(\frac{1}{2}f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}f(x_n)\right)\Delta x$$

Trapezoid Rule 2

Diagram for Trapezoid Rule: Note that the trapezoid rule has a similar accuracy has the **Midpoint Rule**



Trapezoid Rule 3

Trapezoid Rule: Use illustration above

$$f(x) = x^3 - 6x^2 + 9x + 2 \qquad \text{for} \quad x \in [0,5]$$

- The interval $[0,5]$ is divided into 5 subintervals with length $\Delta x = 1$
- Height of the function are evaluated at endpoints of the subintervals
- The **Trapezoid Rule** gives

$$\int_a^b f(x)dx \approx \left(\tfrac{1}{2}f(0) + f(1) + f(2) + f(3) + f(4) + \tfrac{1}{2}f(5)\right)\Delta x$$
$$= \left(\tfrac{1}{2}2 + 6 + 4 + 2 + 6 + \tfrac{1}{2}22\right)\cdot 1 = 30$$

- The actual integral value is **28.75**, so the approximation is 4.3% too high (similar error to the midpoint rule)

Simpson's Rule 1

Simpson's Rule obtains a much more accurate approximation to the integral without having a significantly more complicated formula

- Simpson's rule approximates the function $f(x)$ by quadratics
- The interval of integration $[a,b]$ is divided n subintervals $[x_{i-1},x_i]$
  - Length $\Delta x = \frac{b-a}{n}$
  - The endpoints are $x_0 = a$ and $x_n = b$
  - n must be an even integer
- The formula for **Simpson's rule** is

$$\int_a^b f(x)dx \approx (f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+... +2f(x_{n-2})+4f(x_{n-1})+f(x_n))\frac{\Delta x}{3}$$

Example 1

Example: Use the **Midpoint rule**, **Trapezoid rule**, and **Simpson's rule** to approximate the integral

$$\int_0^2 x^2dx$$

with $n = 4$

Solution: With $n = 4$ the four subintervals are $[0,\frac{1}{2}]$, $[\frac{1}{2},1]$, $[1,\frac{3}{2}]$, and $[\frac{3}{2},2]$, so $\Delta x = \frac{1}{2}$

The midpoints are $c_i = \frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, and $\frac{7}{4}$

Example 2

Solution: With $\Delta x = \frac{1}{2}$, the **Midpoint rule** gives

$$\begin{aligned}\int_0^2 x^2dx &\approx \sum_{i=1}^4 f(c_i)\Delta x \\ &= \sum_{i=1}^4 \left(\frac{i}{2}-\frac{1}{4}\right)^2\frac{1}{2} \\ &= \left(\frac{1+9+25+49}{16}\right)\frac{1}{2} \\ &= \frac{21}{8} = 2.625\end{aligned}$$

Example 3

Solution: With $\Delta x = \frac{1}{2}$, the **Trapezoid rule** gives

$$\begin{aligned}\int_0^2 x^2dx &\approx \left(\tfrac{1}{2}f(x_0)+\textstyle\sum_{i=1}^3 f(x_i)+\tfrac{1}{2}f(x_4)\right)\Delta x \\ &= \left(\tfrac{1}{2}0+\left(\tfrac{1}{2}\right)^2+(1)^2+\left(\tfrac{3}{2}\right)^2+\tfrac{1}{2}(2)^2\right)\tfrac{1}{2} \\ &= 2.75\end{aligned}$$

Example 4

Solution: With $\Delta x = \frac{1}{2}$, **Simpson's rule** gives

$$\begin{aligned}\int_0^2 x^2 dx &\approx \left(f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + f(2)\right)\frac{\Delta x}{3} \\ &= \left(0 + 4\left(\tfrac{1}{2}\right)^2 + 2(1)^2 + 4\left(\tfrac{3}{2}\right)^2 + (2)^2\right)\tfrac{1}{6} \\ &= \tfrac{8}{3}\end{aligned}$$

This is the exact answer. Simpson's rule gives the exact answer for any quadratic.

Example 2 1

Example 2: Consider the function

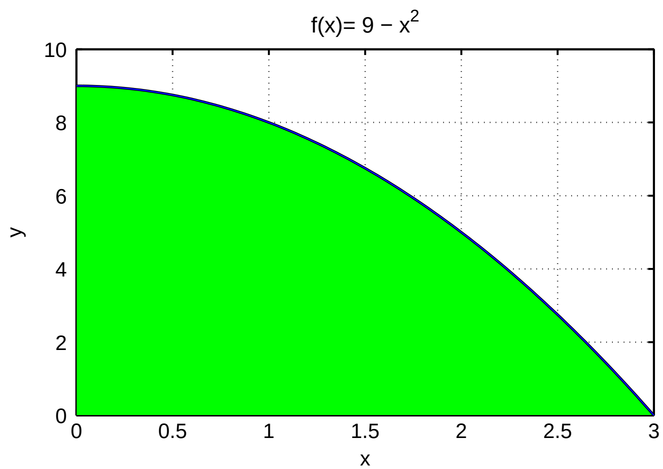
$$f(x) = 9 - x^2$$

- Find the area in the first quadrant under the curve
- Sketch a graph showing the area under the graph
- Use the **Midpoint rule**, **Trapezoid rule**, and **Simpson's rule** to approximate the integral with $n = 6$

Solution: The function intersects the x-axis at $x = 3$

Example 2 2

Solution: Graph of $f(x)$ in the first quadrant



Example 2 3

Solution (cont): The integral defining the area in the previous figure is

$$\int_0^3 (9 - x^2)dx$$

- The integral has limits $x = 0$ and $x = 3$, so with $n = 6$ the subintervals have length, $\Delta x = \frac{1}{2}$
- The midpoints of the subintervals are

$$c_i = \frac{i}{2} - \frac{1}{4} \qquad i = 1, ..., 6$$

Example 2 4

Solution (cont): With $\Delta x = \frac{1}{2}$, the **Midpoint rule** gives

$$\begin{aligned}\int_0^3 (9-x^2)dx &\approx \sum_{i=1}^{6} f(c_i)\Delta x \\ &= \sum_{i=1}^{6}\left(9-\left(\tfrac{i}{2}-\tfrac{1}{4}\right)^2\right)\tfrac{1}{2} \\ &= (8.9375+8.4375+7.4375+5.9375 \\ &\qquad +3.9375+1.4375)\tfrac{1}{2} \\ &= 18.0625\end{aligned}$$

Example 2 5

Solution: With $\Delta x = \frac{1}{2}$ and $x_i = \frac{i}{2}$, the **Trapezoid rule** gives

$$\begin{aligned}\int_0^2 (9-x^2)dx &\approx \left(\tfrac{1}{2}f(0)+\textstyle\sum_{i=1}^{5} f(x_i)+\tfrac{1}{2}f(3)\right)\Delta x \\ &= (4.5+8.75+8+6.75+5+2.75+0)\tfrac{1}{2} \\ &= 17.875\end{aligned}$$

Example 2 6

Solution: With $\Delta x = \frac{1}{2}$, **Simpson's rule** gives

$$\begin{aligned}\int_0^2 (9-x^2)dx &\approx \left(f(0)+4f\left(\tfrac{1}{2}\right)+2f(1)+4f\left(\tfrac{3}{2}\right)+2f(2)\right. \\ &\qquad \left. +4f\left(\tfrac{5}{2}\right)+f(3)\right)\frac{\Delta x}{3} \\ &= (9+4(8.75)+2(8)+4(6.75)+2(5)+4(2.75)+0)\tfrac{1}{6} \\ &= 18\end{aligned}$$

This is the exact answer

Temperature Example 1

Temperature Example: Insects are an important agricultural pest

- Some pesticides have there greatest effects at particular stages of the insect development
- Timing of application of the pesticide can be very significant
- Maturation of insects is often dependent upon temperature more than length of time
- It can be important to track the cumulative temperature rather than the length of time that an insect has been around
- Cumulative temperature T_c (in °C-hr) is found by integrating the temperature $T(t)$ over a period of time

$$T_c = \int_a^b T(t)dt$$

Temperature Example

Temperature Example: Data for temperatures (noon to 7 PM)

Time	12:00	13:00	14:00	15:00	16:00	17:00	18:00	19:00
Temp(°C)	33	34	36	35	32	30	26	24

Use the Trapezoid rule and the data from the table to approximate the cumulative temperature from noon to 7 PM

Note: The average temperature is 31.25 °C

Temperature Example

Solution: Since the length of time between the temperature measurements is one hour, $\Delta t = 1$

The **Trapezoid rule** gives

$$\begin{aligned} T_c &= \int_{12}^{19} T(t)dt \\ &\approx \left(\frac{1}{2}T(12) + \sum_{i=13}^{18} T(i) + \frac{1}{2}T(19) \right) \Delta t \\ &= (16.5 + 34 + 36 + 35 + 32 + 30 + 26 + 12) \cdot 1 \\ &= 221.5 \text{ °C} \cdot \text{hr} \end{aligned}$$

This varies slightly from computing the average temperature and multiplying by the length of time ($31.25 \times 7 = 218.75$)