Calculus for the Life Sciences II

Lecture Notes – Optimization

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Outline

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- These arguments suggest that organisms try to optimize

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- Critical points are often **local minima or maxima** for the function
- This is one application of Calculus, where an **optimal solution** is found

Crow Predation on Whelks

- Sea gulls and crows have learned to feed on various mollusks by dropping their prey on rocks to break the protective shells

Optimal Foraging

Optimal Foraging – Northwestern crows (*Corvus caurinus*) on Mandarte Island

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  - Eat broken whelks

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- **Can this behavior be explained by an optimal foraging decision process?**
- **Is the crow exhibiting a behavior that minimizes its expenditure of energy to feed on whelks?**

Foraging Strategy

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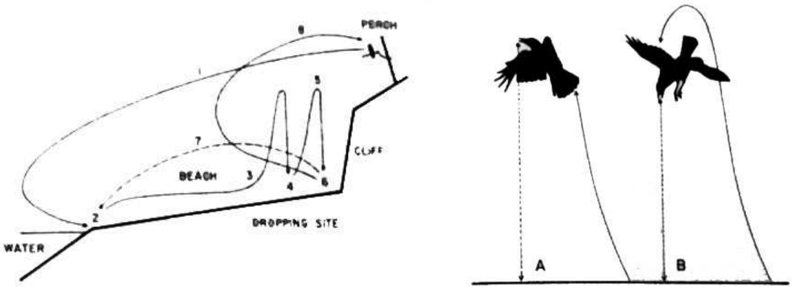

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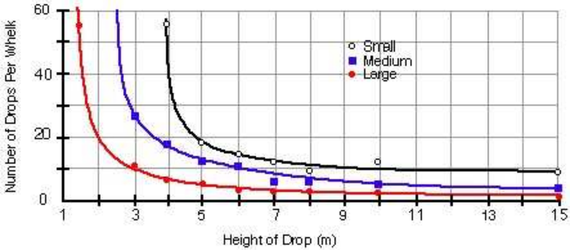
  - Collected and sorted whelks by size
  - Dropped whelks from various heights until they broke
  - Recorded how many drops at each height were required to break each whelk



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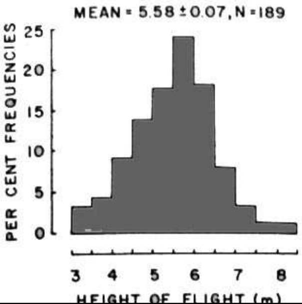
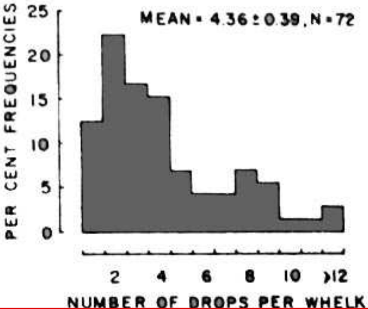
- Crows benefit by selecting the larger ones because they don't need as many drops per whelk, and they gain more energy from consuming a larger one
- Study showed that the whelks broken on the rocks were remarkably similar in size, weighing about 9 grams

Number of Drops

Zach Observation – Height of the drops and number of drops required for many crows to eat whelks used a marked pole on the beach near a favorite dropping location





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- So why do the crows consistently fly to about 5.25 m and use about 4.4 drops to split open a whelk?
- Can this be explained by a mathematical model for minimizing the energy spent, thus supporting an optimal foraging strategy?

Mathematical Model for Energy 1

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  - The energy required to lift the whelk to a certain height and drop it times the number of vertical flights required to split open the whelk
- Concentrate only on this last component of the problem, as it was observed that the crows kept with the same whelk until they broke it open rather than searching for another whelk when one failed to break after a few attempts

Mathematical Model for Energy 2

Energy Function

- The energy is given by the height (H) times the number of drops (N) or
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where k is a constant of proportionality
- Flying higher and increasing the number of drops both increase the use of energy

Mathematical Model for Energy 3

Fitting the Data – Zach's data on dropping large whelks

$H(m)$	1.5	2	3	4	5	6	7	8	10	15
$N(H)$	56	20	10.2	7.6	6	5	4.3	3.8	3.1	2.5

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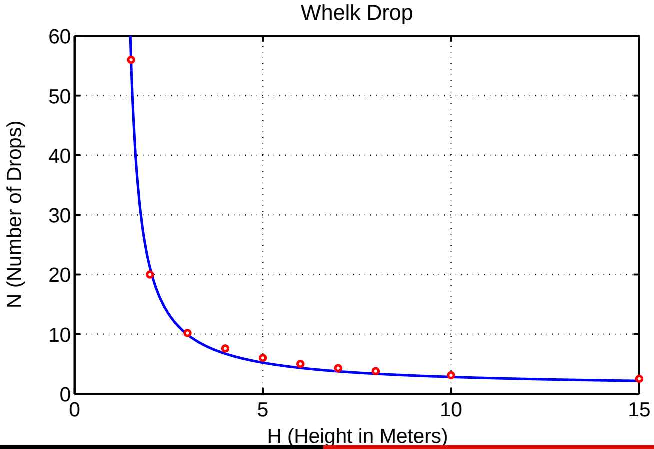
$$N(H) = 1 + \frac{a}{H - b}$$
- The least squares best fit of this function to Zach's data gives $a = 15.97$ and $b = 1.209$

Mathematical Model for Energy 4

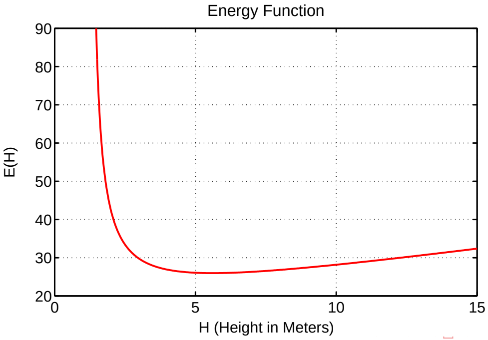
Graph for Whelks being Dropped



Mathematical Model for Energy 5

Graph of Energy Function – The energy function is

$$E(H) = kH\left(1 + \frac{a}{H-b}\right)$$



Mathematical Model for Energy 6

Minimization Problem – Energy satisfies

$$E(H) = kH\left(1 + \frac{a}{H-b}\right)$$

- A **minimum energy** is apparent from the graph with the value around 5.6 m, which is close to the observed value that Zach found the crows to fly when dropping whelks

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- The derivative of $E(H)$ is

$$E'(H) = k\left(1 + \frac{a}{H-b} - \frac{aH}{(H-b)^2}\right) = k\left(\frac{H^2 - 2bH + b^2 - ab}{(H-b)^2}\right)$$

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- The **optimal energy** occurs at the **minimum**, where

$$E'(H) = 0$$

Mathematical Model for Energy

Minimization Problem – The derivative of the Energy function is

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- The numerator is a quadratic with solution

$$H = b \pm \sqrt{ab} = 1.209 \pm 4.394$$

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- Thus, $H = 5.603$ is the **minimum energy**
  ($H = -3.185$ is a maximum, but fails to make sense)
- This computed minimum concurs with the experimental observations, suggesting an **optimal foraging strategy**

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Definition: An **absolute minimum** for a function $f(x)$ occurs at a point $x = c$, if $f(c) < f(x)$ for all x in the domain of f

Optimal Solution

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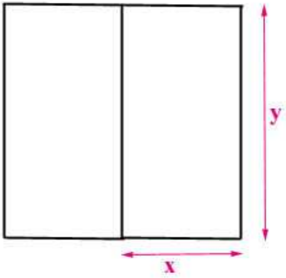
Theorem: Suppose that $f(x)$ is a continuous, differential function on a closed interval $I = [a, b]$, then $f(x)$ achieves its **absolute minimum** (or maximum) on I and its minimum (or maximum) occurs either at a point where $f'(x) = 0$ or at one of the **endpoints of the interval**

Optimal Study Area 1

Optimal Study Area: An ecology student goes into the field with 120 m of string and wants to create two adjacent rectangular study areas with the maximum area possible



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The **Constraint Condition** is the length of string available

$$P(x, y) = 4\,x + 3\,y = 120$$

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Solution (cont): This problem allows the **objective function** of two variables to be reduced by the **constraint condition** to a function of one variable that can readily be **optimized**

Optimal Study Area 3

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- The **constraint condition** is solved for y to give

$$y = \frac{120 - 4\,x}{3}$$

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$$A(x) = x\frac{120 - 4\,x}{3} = 40\,x - \frac{4\,x^2}{3}$$

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- The **domain** of this function is $x \in [0, 30]$

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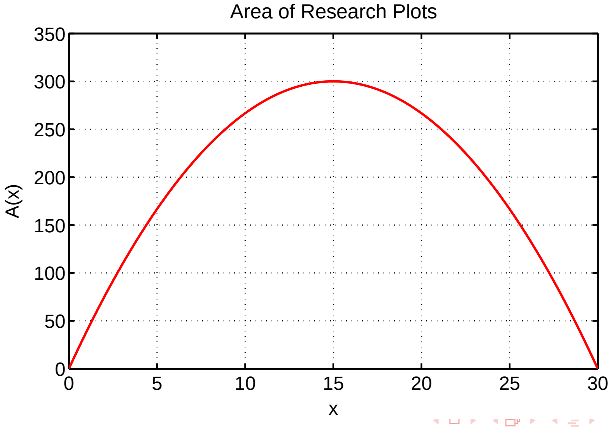
Solution (cont): The **objective function** is a parabola



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- Alternately, we differentiate the **objective function** with

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- The **critical point** occurs when $A'(x_c) = 0$ or $x_c = 15$

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- The **critical point** occurs when $A'(x_c) = 0$ or $x_c = 15$
- The **maximum area** occurs with $x = 15$ m and $y = 20$ m
- To maximize the study areas, the ecology student should make each of the two study areas **15 m wide** and **20 m long** or $A_{max} = $ **300 m²**

Chemical Reaction 1

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- k is the rate constant of the reaction and x is the concentration of X during the reaction
- What is the concentration of X where the reaction rate is at a maximum?

Chemical Reaction 2

Chemical Reaction: Suppose that $k = 50$ (sec^{-1}), $a = 6$ (ppm), and $b = 2$ (ppm), so

$$R(x) = 50(6-x)(2-x) = 50\,x^2 - 400\,x + 600, \quad 0 \le x \le 2$$

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  - At $x = 0$,the reaction rate is $R(0) = 600$ (**maximum**)
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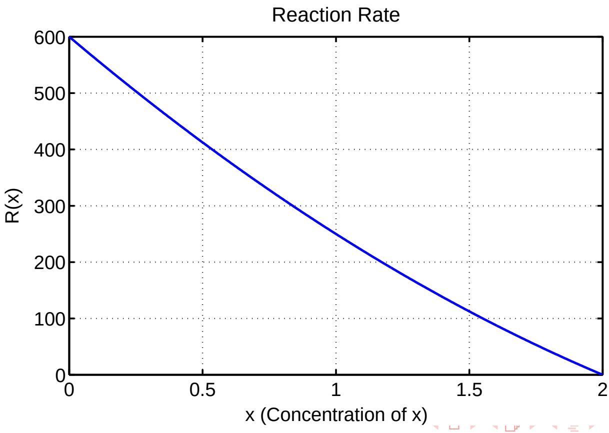
Chemical Reaction

3

Chemical Reaction: Graphing the Reaction Rate



Absolute Extrema of a Polynomial 1

Absolute Extrema of a Polynomial: Consider the cubic polynomial $f(x)$ defined on the interval $x \in [0, 5]$, where

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Find the **absolute extrema** of this polynomial on its **domain**

Skip Example

Absolute Extrema of a Polynomial 2

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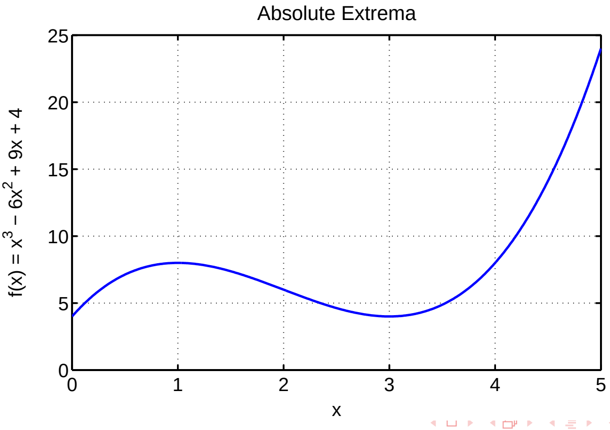
  - $f(3) = 4$ (an **absolute minimum**)
  - $f(5) = 24$ (an **absolute maximum**)

SDSU

Absolute Extrema of a Polynomial

Solution: Graph of cubic polynomial



Crop Yield 1

Example: Crop Yield The yield of an agricultural crop depends on the nitrogen in the soil

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- Find the nitrogen level that produces the maximum crop yield

SDSU

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Solution: Crop yield for $N \geq 0$ satisfies $Y(N) = \frac{N}{1+N^2}$

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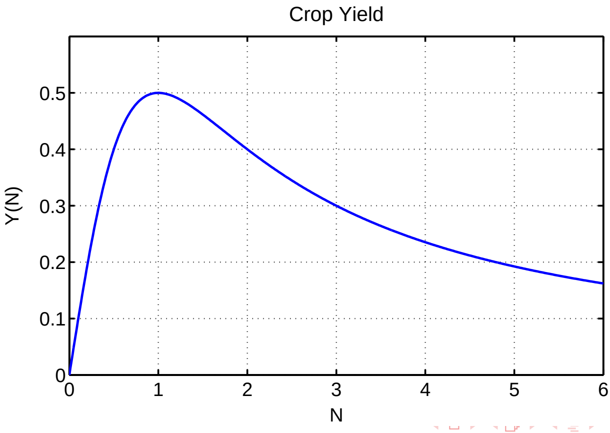
- The endpoints are $N = 0$ and $N \to \infty$
  - $Y(0) = 0$ is an **absolute minimum**
  - As $N \to \infty$, $Y(N) \to 0$, confirming that we found the absolute maximum

Crop Yield 3

Solution: Graph of crop yield function



Wire Problem 1

Example: Wire Problem A wire length L is cut to make a circle and a square

Skip Example

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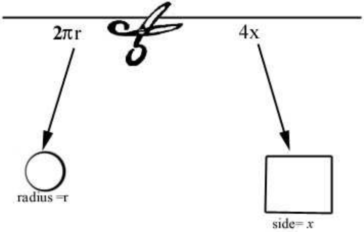
Skip Example

How should the cut be made to maximize the area enclosed by the two shapes?



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From the constraint, r satisfies

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Relative extrema satisfy $A'(x) = 0$, so

$$(4+\pi)x = L$$

Wire Problem

Solution: The **relative extremum** occurs at

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- Cutting the wire at $x = \frac{L}{4+\pi}$ gives the minimum possible area

Wire Problem 5

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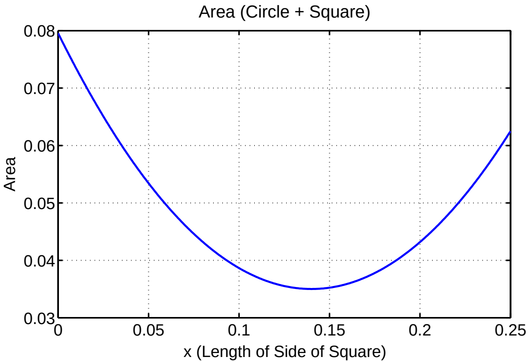
- Since $4\pi < 16$, $A(0) > A\left(\frac{L}{4}\right)$
- The **maximum** occurs when the wire is used to create a circle
- Geometrically, a circle is the most efficient conversion of a linear measurement into area

Wire Problem 6

Solution: Graph of **wire problem** with $L = 1$

$$A(x) = \frac{(1 - 4\,x)^2}{4\pi} + x^2$$



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  - Others proteins are produced at a higher rate as bacteria enter stress due to limitations in some nutrient
  - In stationary phase, bacteria tend to produce all proteins at a significantly lower rate

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Production of a Pharmaceutical: Suppose the production, Q, depends on the population of the bacteria, B,

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- Suppose the population of the bacteria, B, satisfies a **logistic growth curve**

$$B(t) = \frac{2000}{1 + 99\, e^{-0.01t}}$$

Optimal Production of a Pharmaceutical 3

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- The production is at its maximum when $\frac{dQ}{dt} = 0$
- Finding $\frac{dQ}{dt}$ requires the differentiation of a composite function, which uses the chain rule

Optimal Production of a Pharmaceutical 4

Chain Rule Problem: The **composite function** is $Q(B(t))$ with

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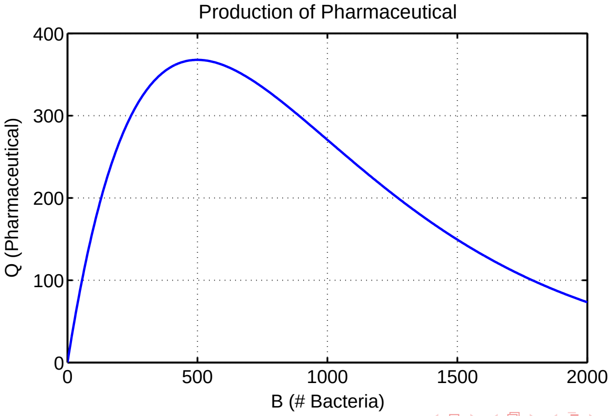
- Notice that this has a maximum at

$$B = 500 \quad \text{with} \quad Q(500) = 1000\,e^{-1} = 367.9$$

Optimal Production of a Pharmaceutical 5

Solution: Graph of $Q(B)$



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Optimal Production of a Pharmaceutical 4

Chain Rule Problem: The **composite function** is $Q(B(t))$ with

$$Q(B) = 2Be^{-0.002B} \quad \text{and} \quad B(t) = \frac{2000}{1 + 99\, e^{-0.01t}}$$

- The **chain rule** for differentiating the composite function is

$$\frac{dQ}{dt} = \frac{dQ}{dB}\frac{dB}{dt} = Q'(B)B'(t)$$

- The **derivative** of $Q(B)$ is

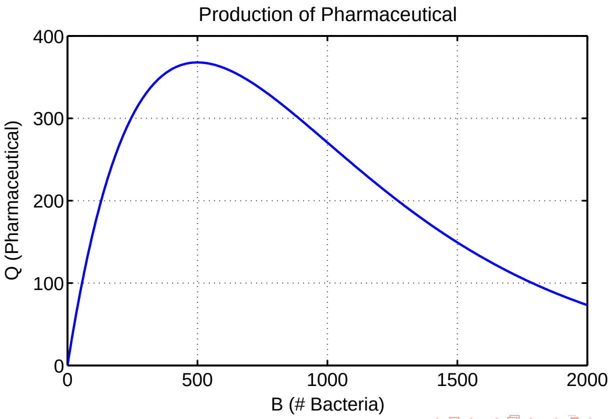
$$Q'(B) = 2e^{-0.002B}(1 - 0.002B)$$

- Notice that this has a maximum at

$$B = 500 \quad \text{with} \quad Q(500) = 1000\, e^{-1} = 367.9$$

Optimal Production of a Pharmaceutical 5

Solution: Graph of $Q(B)$



Optimal Production of a Pharmaceutical 6

Solution: The **bacterial population**, $B(t)$, is

$$B(t) = \frac{2000}{1 + 99\,e^{-0.01t}} = 2000\left(1 + 99\,e^{-0.01t}\right)^{-1}$$

Optimal Production of a Pharmaceutical 6

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Optimal Production of a Pharmaceutical 6

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Optimal Production of a Pharmaceutical 6

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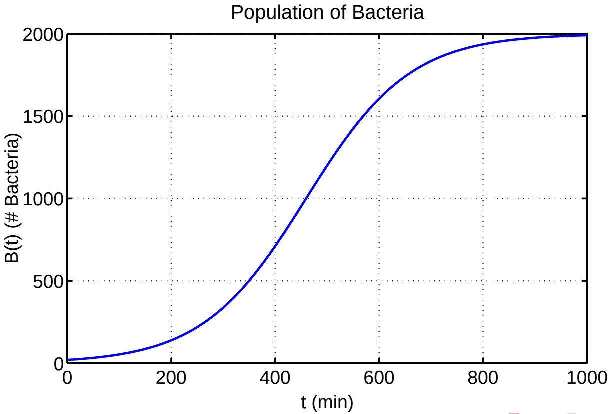
- This function is always **positive** or **constantly increasing**
- It increases at different rates with varying times

Optimal Production of a Pharmaceutical

Solution: Graph of $B(t)$



Optimal Production of a Pharmaceutical

Solution: The derivative of the composite function is

$$\frac{dQ}{dt} = Q'(B(t))B'(t) = \frac{3960e^{-0.002B}(1 - 0.002\,B)e^{-0.01t}}{(1 + 99\,e^{-0.01t})^2}$$

Optimal Production of a Pharmaceutical 8

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Optimal Production of a Pharmaceutical 8

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- The only **critical point** occurs when $B = 500$
- We solve $B(t) = 500$

$$\begin{aligned} \frac{2000}{1+99\,e^{-0.01t}} &= 500 \\ 1+99\,e^{-0.01t} &= 4 \\ e^{0.01t} &= 33 \\ t &= 100\ln(33) = 349.65 \text{ min} \end{aligned}$$

Optimal Production of a Pharmaceutical 9

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Optimal Production of a Pharmaceutical 9

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This is the **optimal production of the pharmaceutical**