Calculus for the Life Sciences II

Lecture Notes – Numerical Methods for Differential Equations

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Outline

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Introduction

Introduction

- Differential Equations provide useful models
- Realistic Models are often Complex
- Most differential equations can **not** be solved exactly
- Develop numerical methods to solve differential equations

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Pollution in a Lake   1

Pollution in a Lake

- Previously studied a simple model for **Lake Pollution**
- Complicate by adding time-varying pollution source
- Include periodic flow for seasonal effects
- Present numerical method to simulate the model

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Pollution in a Lake 2

Non-point Source of Pollution and Seasonal Flow Variation

- Consider a non-point source, such as agricultural runoff of pesticide
  - Assume a pesticide is removed from the market
  - If the pesticide doesn't degrade, it leaches into runoff water
- Concentration of the pesticide in the river being time-varying
- Typically, there is an exponential decay after the use of the pesticide is stopped
  - Example of concentration

$$p(t) = 5\, e^{-0.002t}$$

Pollution in a Lake 3

Including Seasonal Effects

- River flows vary seasonally
- Assume lake maintains a constant volume, V
- Seasonal flow (time varying) entering is reflected with same outflowing flow
  - Example of sinusoidal annual flow

$$f(t) = 100 + 50\cos(0.0172t)$$

Pollution in a Lake 4

Mathematical Model: Use Mass Balance

**The change in amount of pollutant =
Amount entering - Amount leaving**

- Amount entering is concentration of the pollutant in the river times the flow rate of the river

$$f(t)p(t)$$

- Assume the lake is well-mixed
- Amount leaving is concentration of the pollutant in the lake times the flow rate of the river

$$f(t)c(t)$$

- The amount of pollutant in the lake, $a(t)$, satisfies

$$\frac{da}{dt} = f(t)p(t) - f(t)c(t)$$

Pollution in a Lake 5

Mathematical Model: Let the concentration be $c(t) = \frac{a(t)}{V}$

$$\frac{dc(t)}{dt} = \frac{f(t)}{V}(p(t) - c(t)) \quad \text{with} \quad c(0) = c_0$$

- Assume that the volume of the lake is 10,000 m^3 and the initial level of pollutant in the lake is $c_0 = 5$ ppm
- With $p(t)$ and $f(t)$ fom before, model is

$$\frac{dc(t)}{dt} = (0.01 + 0.005\cos(0.0172t))(5\, e^{-0.002t} - c(t))$$

- Complicated, but an exact solution exists
- Show an easier numerical method to approximate the solution

Slide 1

Introduction
Euler's Method
Improved Euler's Method

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Euler's Method

1

Initial Value Problem: Consider

$$\frac{dy}{dt} = f(t, y) \quad \text{with} \quad y(t_0) = y_0$$

- From the definition of the derivative

$$\frac{dy}{dt} = \lim_{h \to 0} \frac{y(t + h) - y(t)}{h}$$

- Instead of taking the limit, fix h, so

$$\frac{dy}{dt} \approx \frac{y(t + h) - y(t)}{h}$$

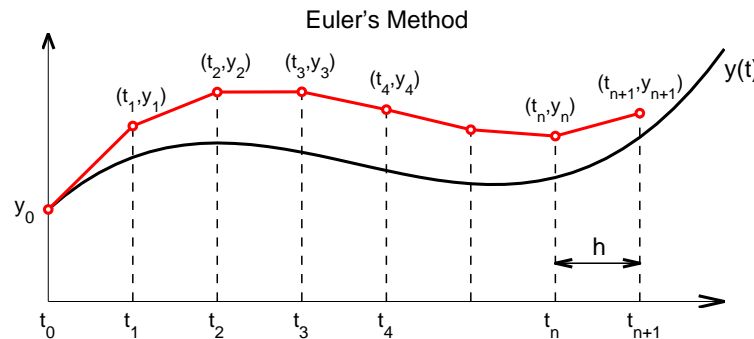- Substitute into the differential equation and with algebra write

$$y(t + h) \approx y(t) + h f(t, y)$$

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Slide 2

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Euler's Method

2

Euler's Method for a fixed h is

$$y(t + h) = y(t) + h f(t, y)$$

- Geometrically, Euler's method looks at the slope of the tangent line
  - The approximate solution follows the tangent line for a time step h
  - Repeat this process at each time step to obtain an approximation to the solution
- The ability of this method to track the solution accurately depends on the length of the time step, h, and the nature of the function $f(t, y)$
- This technique is rarely used as it has very bad convergence properties to the actual solution

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Slide 3

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Euler's Method

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Graph of Euler's Method

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Euler's Method

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Euler's Method Formula: Euler's method is just a discrete dynamical system for approximating the solution of a continuous model

- Let $t_{n+1} = t_n + h$
- Define $y_n = y(t_n)$
- The initial condition gives $y(t_0) = y_0$
- **Euler's Method** is the discrete dynamical system

$$y_{n+1} = y_n + h\, f(t_n, y_n)$$

- Euler's Method only needs the initial condition to start and the right hand side of the differential equation (the **slope field**), $f(t, y)$ to obtain the approximate solution

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Malthusian Growth Example   1

Malthusian Growth Example: Consider the model

$$\frac{dP}{dt} = 0.2\,P \qquad \text{with} \qquad P(0) = 50$$

Find the exact solution and approximate the solution with Euler's Method for $t \in [0, 1]$ with $h = 0.1$

Solution: The exact solution is

$$P(t) = 50\,e^{0.2t}$$

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Malthusian Growth Example   2

Solution (cont): The **Formula for Euler's Method** is

$$P_{n+1} = P_n + h\,0.2\,P_n$$

The initial condition $P(0) = 50$ implies that $t_0 = 0$ and $P_0 = 50$
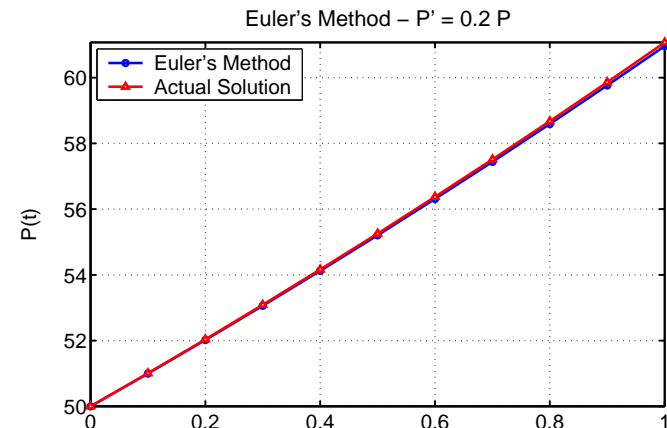
Create a table for the Euler iterates

t_n	P_n
$t_0 = 0$	$P_0 = 50$
$t_1 = t_0 + h = 0.1$	$P_1 = P_0 + 0.1(0.2P_0) = 50 + 1 = 51$
$t_2 = t_1 + h = 0.2$	$P_2 = P_1 + 0.1(0.2P_1) = 51 + 1.02 = 52.02$
$t_3 = t_2 + h = 0.3$	$P_3 = P_2 + 0.1(0.2P_2) = 52.02 + 1.0404 = 53.0604$

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Malthusian Growth Example   3

Solution (cont): Iterations are easily continued - Below is table of the actual solution and the Euler's method iterates

t	Euler Solution	Actual Solution
0	50	50
0.1	51	51.01
0.2	52.02	52.041
0.3	53.060	53.092
0.4	54.122	54.164
0.5	55.204	55.259
0.6	56.308	56.375
0.7	57.434	57.514
0.8	58.583	58.676
0.9	59.755	59.861
1.0	60.950	61.070

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Malthusian Growth Example   4

Graph of Euler's Method for Malthusian Growth Example

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Malthusian Growth Example — 5

Error Analysis and Larger Stepsize

- The table and the graph shows that Euler's method is tracking the solution fairly well over the interval of the simulation
- The error at $t = 1$ is only 0.2%
- However, this is a fairly short period of time and the stepsize is relatively small
- What happens when the stepsize is increased and the interval of time being considered is larger?

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Malthusian Growth Example — 6

Graph of Euler's Method with $h = 0.5$

There is a 9% error in the numerical solution at $t = 10$

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Euler's Method with $f(t, y)$ — 1

Euler's Method with $f(t, y)$**:** Consider the model

$$\frac{dy}{dt} = y + t \qquad \text{with} \qquad y(0) = 3$$

Find the approximate solution with Euler's Method at $t = 1$ with stepsize $h = 0.25$
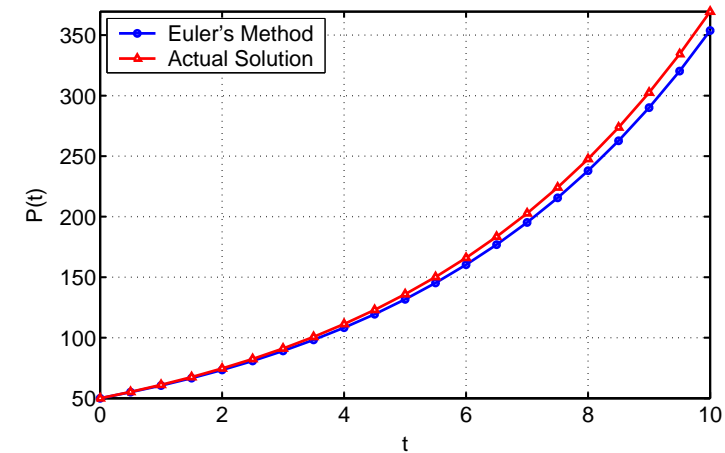
Compare the Euler solution to the exact solution

$$y(t) = 4\,e^t - t - 1$$

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Euler's Method with $f(t, y)$ — 2

Solution: Verify the actual solution:

1. Initial condition:

$$y(0) = 4\,e^0 - 0 - 1 = 3$$

2. The differential equation:

$$\frac{dy}{dt} = 4\,e^t - 1$$
$$y(t) + t = 4\,e^t - t - 1 + 1 = 4\,e^t - 1$$

Euler's formula for this problem is

$$y_{n+1} = y_n + h(y_n + t_n)$$

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Euler's Method with $f(t,y)$ 3

Solution (cont): Euler's formula with $h = 0.25$ is

$$y_{n+1} = y_n + 0.25(y_n + t_n)$$

t_n	Euler solution y_n
$t_0 = 0$	$y_0 = 3$
$t_1 = 0.25$	$y_1 = y_0 + h(y_0 + t_0) = 3 + 0.25(3 + 0) = 3.75$
$t_2 = 0.5$	$y_2 = y_1 + h(y_1 + t_1) = 3.75 + 0.25(3.75 + 0.25) = 4.75$
$t_3 = 0.75$	$y_3 = y_2 + h(y_2 + t_2) = 4.75 + 0.25(4.75 + 0.5) = 6.0624$
$t_4 = 1$	$y_4 = y_3 + h(y_3 + t_3) = 6.0624 + 0.25(6.0624 + 0.75) = 7.7656$

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Euler's Method with $f(t,y)$ 4

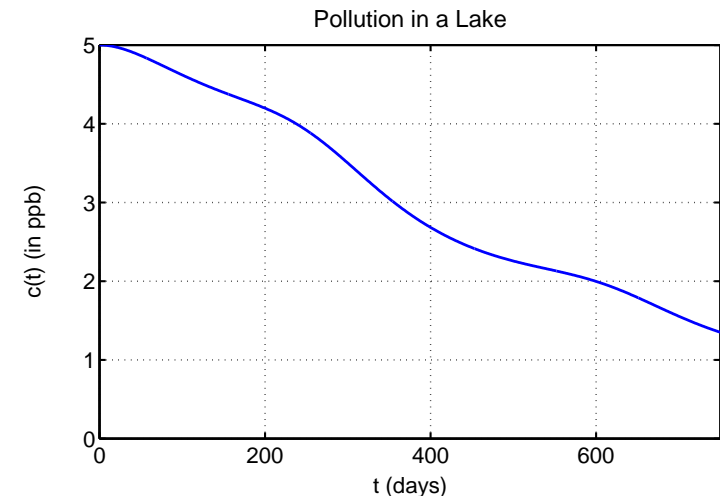**Solution (cont): Error Analysis**

- $y_4 = 7.7656$ corresponds to the approximate solution of $y(1)$
- The actual solution gives $y(1) = 8.87312$, so the Euler approximation with this large stepsize is not a very good approximation of the actual solution with a 12.5% error
- If the stepsize is reduced to $h = 0.1$, then Euler's method requires 10 steps to find an approximate solution for $y(1)$
- It can be shown that the Euler approximate of $y(1)$, $y_{10} = 8.37497$, which is better, but still has a 5.6% error

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Numerical Solution of the Lake Problem   1

Numerical Solution of the Lake Problem Earlier described a more complicated model for pollution entering a lake with an oscillatory flow rate and an exponentially falling concentration of the pollutant entering the lake via the river

- The initial value problem with $c(0) = 5 = c_0$

$$\frac{dc}{dt} = (0.01 + 0.005 \cos(0.0172t))(5\,e^{-0.002t} - c(t))$$

- The **Euler's formula** is

$$c_{n+1} = c_n + h(0.01 + 0.005 \cos(0.0172t_n))(5\,e^{-0.002t_n} - c_n)$$

- The model was simulated for 750 days with $h = 1$

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Numerical Solution of the Lake Problem   2

Graph of Simulation

Pollution in a Lake

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Numerical Solution of the Lake Problem   3

Simulation: This solution shows a much more complicated behavior for the dynamics of the pollutant concentration in the lake

- Could you have predicted this behavior or determined quantitative results, such as when the pollution level dropped below 2 ppm?
- This example is much more typical of what we might expect from more realistic biological problems
- The numerical methods allow the examination of more complex situations, which allows the scientist to consider more options in probing a given situation
- Euler's method for this problem traces the actual solution very well, but better numerical methods are usually used   **SDSU**

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Euler Example A   1

Euler Example A: Consider the initial value problem

$$\frac{dy}{dt} = -2\,y^2 \qquad \text{with} \qquad y(0) = 2$$

Skip Example

- With a stepsize of $h = 0.2$, use Euler's method to approximate $y(t)$ at $t = 1$
- Show that the actual solution of this problem is

$$y(t) = \frac{2}{4\,t + 1}$$

- Determine the percent error between the approximate solution and the actual solution at $t = 1$ **SDSU**

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Euler Example A   2

Solution: Euler's formula with $h = 0.2$ for this example is

$$y_{n+1} = y_n - h(2y_n^2) = y_n - 0.4\,y_n^2$$

t_n	y_n
$t_0 = 0$	$y_0 = 2$
$t_1 = t_0 + h = 0.2$	$y_1 = y_0 - 0.4y_0^2 = 2 - 0.4(4) = 0.4$
$t_2 = t_1 + h = 0.4$	$y_2 = y_1 - 0.4y_1^2 = -0.4 - 0.4(0.16) = 0.336$
$t_3 = t_2 + h = 0.6$	$y_3 = y_2 - 0.4y_2^2 = 0.336 - 0.4(0.1129) = 0.2908$
$t_4 = t_3 + h = 0.8$	$y_4 = y_3 - 0.4y_3^2 = 0.2908 - 0.4(0.08459) = 0.2570$
$t_5 = t_4 + h = 1.0$	$y_5 = y_4 - 0.4y_4^2 = 0.2570 - 0.4(0.06605) = 0.2306$

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Euler Example A   3

Solution (cont): Verify that the solution is

$$y(t) = \frac{2}{4\,t + 1} = 2(4\,t + 1)^{-1}$$

- Compute the derivative

$$\frac{dy}{dt} = -2(4\,t + 1)^{-2}(4) = -8(4\,t + 1)^{-2}$$

- However, $-2\,(y(t))^2 = -2(2(4\,t + 1)^{-1})^2 = -8(4\,t + 1)^{-2}$
- Thus, the differential equation is satisfied by the solution that is given
- At $t = 1$, $y(1) = 0.4$
- The percent error is

$$100 \times \frac{y_{Euler}(1) - y_{actual}}{y_{actual}(1)} = \frac{100(0.2306 - 0.4)}{0.4} = -42.4\%$$

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Euler Example B 1

Euler Example B: Consider the initial value problem

$$\frac{dy}{dt} = 2\frac{t}{y} \qquad \text{with} \qquad y(0) = 2$$

- With a stepsize of $h = 0.25$, use Euler's method to approximate $y(t)$ at $t = 1$
- Show that the actual solution of this problem is

$$y(t) = \sqrt{2\,t^2 + 4}$$

- Determine the percent error between the approximate solution and the actual solution at $t = 1$

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Euler Example B 2

Solution: Euler's formula with $h = 0.25$ for this example is

$$y_{n+1} = y_n + h\left(\frac{2t_n}{y_n}\right) = y_n + 0.5\left(\frac{t_n}{y_n}\right)$$

t_n	y_n
$t_0 = 0$	$y_0 = 2$
$t_1 = t_0 + h = 0.25$	$y_1 = y_0 + 0.5t_0/y_0 = 2 + 0.5(0/2) = 2$
$t_2 = t_1 + h = 0.5$	$y_2 = y_1 + 0.5t_1/y_1 = 2 + 0.5(0.25/2) = 2.0625$
$t_3 = t_2 + h = 0.75$	$y_3 = y_2 + 0.5t_2/y_2 = 2.0625 + 0.5(0.5/2) = 2.1875$
$t_4 = t_3 + h = 1.0$	$y_4 = y_3 + 0.5t_3/y_3 = 2.1875 + 0.5(0.75/2) = 2.375$

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Euler Example B 3

Solution (cont): Verify that the solution is
$$y(t) = (2t^2 + 4)^{0.5}$$

- Compute the derivative

$$\frac{dy}{dt} = 0.5(2t^2 + 4)^{-0.5}(4t) = 2t(2t^2 + 4)^{-0.5}$$

- However, $2t/y(t) = 2t/(2t^2 + 4)^{0.5} = 2t(2t^2 + 4)^{-0.5}$
- Thus, the differential equation is satisfied by the solution that is given
- At $t = 1$, $y(1) = \sqrt{6} = 2.4495$
- The percent error is

$$100 \times \frac{y_{Euler}(1) - y_{actual}}{y_{actual}(1)} = \frac{100(2.375 - 2.4495)}{2.4495} = -3.04\%$$

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Time-varying Population Model 1

Time-varying Population Model: A Malthusian growth model with a time-varying growth rate is

$$\frac{dP}{dt} = (0.2 - 0.02\,t)P \qquad \text{with} \qquad P(0) = 5000$$

- With a stepsize of $h = 0.2$, use Euler's method to approximate $P(t)$ at $t = 1$
- Show that the actual solution of this problem is

$$P(t) = 5000\,e^{0.2t - 0.01t^2}$$

- Determine the percent error between the approximate solution and the actual solution at $t = 1$
- Use the actual solution to find the maximum population of this growth model and when it occurs

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Time-varying Population Model 2

Solution: Euler's formula with $h = 0.2$ for this example is

$$P_{n+1} = P_n + h(0.2 - 0.02t_n)P_n$$

t_n	P_n
$t_0 = 0$	$P_0 = 5000$
$t_1 = t_0 + h = 0.2$	$P_1 = P_0 + 0.2(0.2 - 0.02t_0)P_0 = 5200$
$t_2 = t_1 + h = 0.4$	$P_2 = P_1 + 0.2(0.2 - 0.02t_1)P_1 = 5403.8$
$t_3 = t_2 + h = 0.6$	$P_3 = P_2 + 0.2(0.2 - 0.02t_2)P_2 = 5611.35$
$t_4 = t_3 + h = 0.8$	$P_4 = P_3 + 0.2(0.2 - 0.02t_3)P_3 = 5822.3$
$t_5 = t_4 + h = 1.0$	$P_5 = P_4 + 0.2(0.2 - 0.02t_4)P_4 = 6036.6$

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Time-varying Population Model 3

Solution (cont): Verify that the solution is

$$P(t) = 5000\, e^{0.2t - 0.01t^2}$$

- Compute the derivative

$$\frac{dP}{dt} = 5000\, e^{0.2t - 0.01t^2}(0.2 - 0.02\, t)$$

- However, $(0.2 - 0.02\, t)P(t) = 5000\, e^{0.2t - 0.01t^2}(0.2 - 0.02\, t)$
- Thus, the differential equation is satisfied by the solution that is given
- At $t = 1$, $P(1) = 6046.2$
- The percent error is

$$100 \times \frac{P_{Euler}(1) - P_{actual}}{P_{actual}(1)} = \frac{100(6036.6 - 6046.2)}{6046.2} = -0.16\%$$

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Time-varying Population Model 4

Solution (cont): Maximum of the population

- The **maximum** is when the derivative is equal to zero
- Because $P(t)$ is positive, the derivative is zero (growth rate falls to zero) when $0.2 - 0.02\, t = 0$ or $t = 10$ years
- This is substituted into the actual solution

$$P(10) = 5000\, e^1 = 13,591.4$$

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Improved Euler's Method 1

Improved Euler's Method: There are many techniques to improve the **numerical solutions of differential equations**

- Euler's Method is simple and intuitive, but lacks accuracy
- Numerical methods are available through standard software, like Maple or MatLab
- Some of the best are a class of single step methods called **Runge-Kutta methods**
- The simplest of these is called the Improved Euler's method
- Showing why this technique is significantly better than Euler's method is beyond the scope of this course

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Improved Euler's Method 2

Improved Euler's Method Formula: This technique is an easy extension of Euler's Method

- The Improved Euler's method uses an average of the Euler's method and an Euler's method approximation to the function
- Let $y(t_0) = y_0$ and define $t_{n+1} = t_n + h$ and the approximation of $y(t_n)$ as y_n
- First approximate y by Euler's method, so define

$$ye_n = y_n + h\, f(t_n, y_n)$$

- The Improved Euler's formula starts with $y(t_0) = y_0$ and becomes the discrete dynamical system

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_n + h, ye_n)\right)$$

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Example: Improved Euler's Method 1

Example: Improved Euler's Method: Consider the initial value problem:

$$\frac{dy}{dt} = y + t \qquad \text{with} \qquad y(0) = 3$$

- The solution to this differential equation is

$$y(t) = 4\,e^t - t - 1$$

- Numerically solve this using Euler's Method and Improved Euler's Method using $h = 0.1$
- Compare these numerical solutions

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Example: Improved Euler's Method 2

Solution: Let $y_0 = 3$, the Euler's formula is

$$y_{n+1} = y_n + h(y_n + t_n) = y_n + 0.1(y_n + t_n)$$

The Improved Euler's formula is

$$ye_n = y_n + h(y_n + t_n) = y_n + 0.1(y_n + t_n)$$

with

$$y_{n+1} = y_n + \tfrac{h}{2}\left((y_n + t_n) + (ye_n + t_n + h)\right)$$
$$y_{n+1} = y_n + 0.05\left(y_n + ye_n + 2\,t_n + 0.1\right)$$

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Example: Improved Euler's Method 3

Solution: Below is a table of the numerical computations

t	Euler's Method	Improved Euler	Actual
0	$y_0 = 3$	$y_0 = 3$	$y(0) = 3$
0.1	$y_1 = 3.3$	$y_1 = 3.32$	$y(0.1) = 3.3207$
0.2	$y_2 = 3.64$	$y_2 = 3.6841$	$y(0.2) = 3.6856$
0.3	$y_3 = 4.024$	$y_3 = 4.0969$	$y(0.3) = 4.0994$
0.4	$y_4 = 4.4564$	$y_4 = 4.5636$	$y(0.4) = 4.5673$
0.5	$y_5 = 4.9420$	$y_5 = 5.0898$	$y(0.5) = 5.0949$
0.6	$y_6 = 5.4862$	$y_6 = 5.6817$	$y(0.6) = 5.6885$
0.7	$y_7 = 6.0949$	$y_7 = 6.3463$	$y(0.7) = 6.3550$
0.8	$y_8 = 6.7744$	$y_8 = 7.0912$	$y(0.8) = 7.1022$
0.9	$y_9 = 7.5318$	$y_9 = 7.9247$	$y(0.9) = 7.9384$
1	$y_{10} = 8.3750$	$y_{10} = 8.8563$	$y(1) = 8.8731$

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Example: Improved Euler's Method 4

Solution: Comparison of the numerical simulations

- It is very clear that the Improved Euler's method does a substantially better job of tracking the actual solution
- The Improved Euler's method requires only one additional function, $f(t, y)$, evaluation for this improved accuracy
- At $t = 1$, the Euler's method has a -5.6% error from the actual solution
- At $t = 1$, the Improved Euler's method has a -0.19% error from the actual solution

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