Calculus for the Life Sciences II

Lecture Notes – Nonlinear Dynamical Systems

Joseph M. Mahaffy,
⟨mahaffy@math.sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://www-rohan.sdsu.edu/~jmahaffy

Fall 2012

Outline

Introduction

Discrete Growth Models

- The Discrete Malthusian growth model shows exponential growth
- Most animal populations grow exponentially soon after settling
- With population growth, crowding pressure decreases the growth rate
  - Space and resource limitation
  - Toxic build up

Yeast Study 1

Growing Culture of Yeast: Classic study by Carlson in 1913

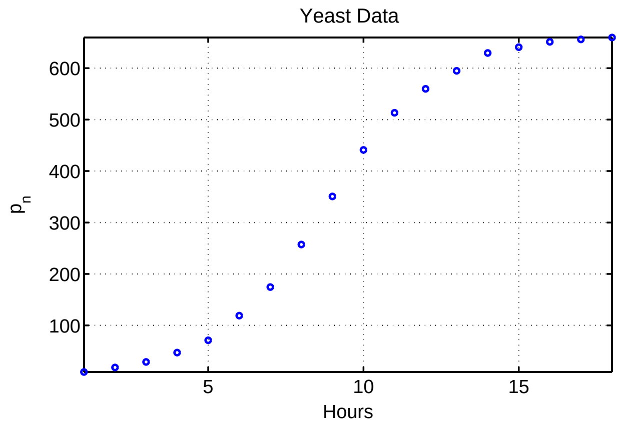
Time	Population	Time	Population	Time	Population
1	9.6	7	174.6	13	594.8
2	18.3	8	257.3	14	629.4
3	29.0	9	350.7	15	640.8
4	47.2	10	441.0	16	651.1
5	71.1	11	513.3	17	655.9
6	119.1	12	559.7	18	659.6

These data show a classic **S-shape curve**

[1] T. Carlson Über Geschwindigkeit und Grösse der Hefevermehrung in Würze. *Biochem. Z.* (1913) **57**, 313–334

Yeast Study 2

Carlson (1913) Yeast data: Classic **S-shape curve** with initial accelerating growth, then eventually saturation



Discrete Growth Models 1

Discrete Dynamical Growth Model

There are two standard forms for **discrete population models**

One form uses a **growth function**, $G(p_n)$

$$p_{n+1} = p_n + G(p_n)$$

The population at the next time interval $(n+1)$ equals the population at the current time interval (n) plus the net growth of the current population, $G(p_n)$

Discrete Growth Models 2

Discrete Dynamical Model with Updating Function

A more general form satisfies

$$p_{n+1} = F(p_n)$$

- An **iterative map** – the population at the $(n+1)^{st}$ generation depends on the population at the n^{th} generation
- The function $F(p)$ is called the **updating function**
- The graph of the updating function
  - The $(n+1)^{st}$ generation is on the vertical axis
  - The n^{th} generation is on the horizontal axis
  - Usually want **identity map** to find **equilibria**

Logistic Growth Model 1

Logistic Growth Model

- Malthusian growth uses a linear updating function and grows exponentially without bound
- Most populations have a decreasing growth rate due to crowding effects
- Easiest form is to insert a quadratic term (negative) to the updating function
- This is the **Logistic Growth model**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

- This equation has the Malthusian growth model with the additional term $-rp_n^2/M$
- The parameter M is called the **carrying capacity** of the population

Logistic Growth Model 2

Behavior of the Logistic Growth Model

- The Logistic growth model shows complicated dynamics – shown by ecologist May (1974)
- There is **no exact solution** to this discrete dynamical system
- Given the **Logistic Growth model**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

  - There are **equilibria** at 0 and M
  - The parameter r has restricted values ($r < 3$) with more complex behavior for higher values of r
- Numerous applets available on the web to view behavior

Yeast Study 1

Logistic Growth Model for Carlson Yeast Study

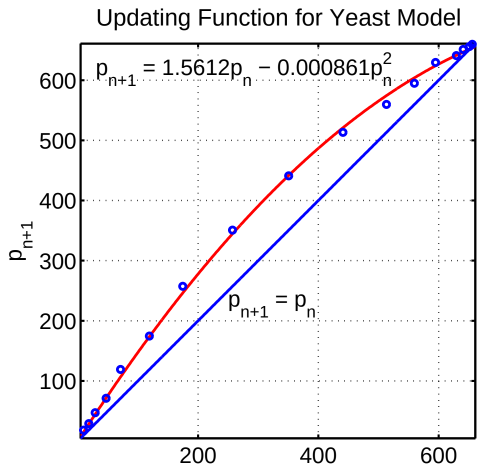
- **Logistic Growth model** has form

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

- Use successive data values to obtain p_{n+1} and p_n
- The first two points are $(9.6, 18.3)$ and $(18.3, 29.0)$ with others found similarly
- The graph of the data is fit with the best quadratic passing through the origin

Yeast Study 2

Updating Function: Graph of best fitting **quadratic** through the origin of data, p_{n+1} vs p_n, and the identity function



Yeast Study 3

- Recall the **logistic growth model** has the form

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

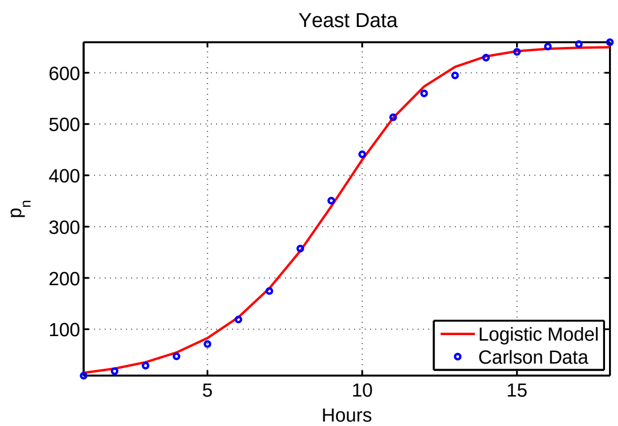
- The best fitting model to the yeast data is

$$p_{n+1} = 1.5612\,p_n - 0.000861\,p_n^2$$

- It follows that $r = 0.5612$ and $M = 650.4$

Yeast Study 4

Simulation: The model is easily simulated and by varying the initial population to $p_1 = 15.0$, a best fit to the data is found



Equilibria

Consider the **general discrete dynamical model:**

$$p_{n+1} = F(p_n)$$

Study the **qualitative behavior of discrete dynamical equations**

- The **first step in any analysis** is finding **equilibria**
- This is simply an **algebraic equation**
- An equilibrium point of a discrete dynamical system is where there is no change in the variable from one iteration to the next
- Mathematically, $p_e = F(p_e)$
- Geometrically, this is when $F(p)$ crosses the **identity map**

Equilibria for Logistic Growth Model

Consider the **logistic growth model:**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

If $r > 0$, then equilibria satisfy

$$\begin{aligned} p_e &= p_e + rp_e\left(1 - \frac{p_e}{M}\right) \\ rp_e\left(1 - \frac{p_e}{M}\right) &= 0 \end{aligned}$$

Thus, $p_e = 0$ or $p_e = M$

The equilibria for the Logistic growth model are either

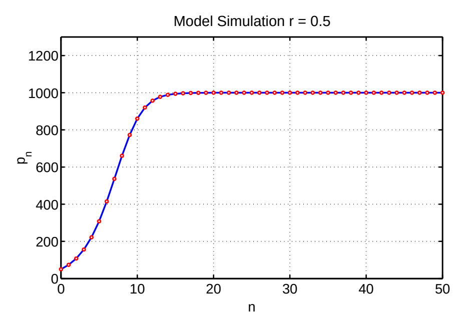
- The **trivial solution** $p_e = 0$ (no population) or
- The **carrying capacity** $p_e = M$

Logistic Growth Model Simulation 1

Consider the **logistic growth model:**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

Let $p_0 = 50$, $M = 1000$, and $r = 0.5$



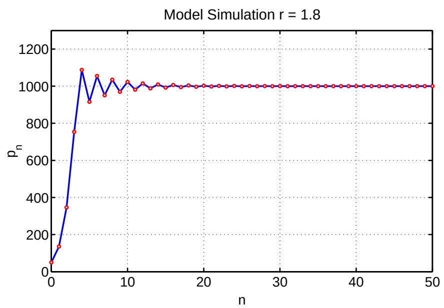
Simulation monotonically approaches carrying capacity

Logistic Growth Model Simulation 2

Consider the **logistic growth model:**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

Let $p_0 = 50$, $M = 1000$, and $r = 1.8$



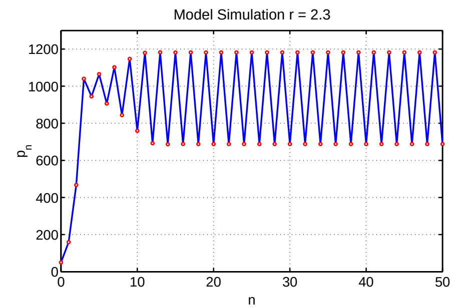
Simulation oscillates, but approaches carrying capacity

Logistic Growth Model Simulation 3

Consider the **logistic growth model:**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

Let $p_0 = 50$, $M = 1000$, and $r = 2.3$



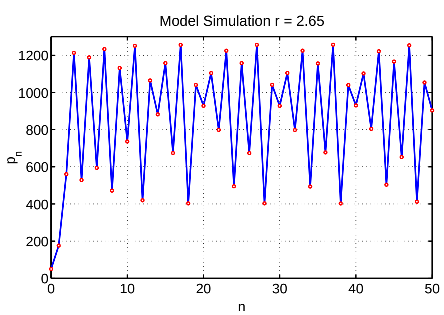
Simulation oscillates with period 2 about carrying capacity

Logistic Growth Model Simulation 4

Consider the **logistic growth model:**

$$p_{n+1} = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

Let $p_0 = 50$, $M = 1000$, and $r = 2.65$



Simulation is chaotic with unpredictable results

Stability of Logistic Growth Model 1

Stability of Logistic Growth Model

- Equilibria are easy to find, but behavior of the model varies dramatically as shown by simulations above
- There are mathematical tools that help predict some of these behaviors
- The discrete logistic growth model is

$$p_{n+1} = f(p_n) = p_n + rp_n\left(1 - \frac{p_n}{M}\right)$$

- The derivative of the function $f(p)$ is valuable for determining the behavior of the discrete dynamical system near an equilibrium point

Stability of Logistic Growth Model

- The **Equilibria** are

$$p_e = 0 \quad \text{and} \quad p_e = M$$

- The **derivative** of $f(p) = (1+r)p - rp^2/M$ is

$$f'(p) = 1 + r - \frac{2rp}{M}$$

- Evaluation of the derivative at the equilibria gives some information about the **behavior of the discrete dynamical model**

Stability of Logistic Growth Model

Consider the **Trivial Equilibrium**, $p_e = 0$

- Since the **derivative** is

$$f'(p) = 1 + r - \frac{2rp}{M}$$

- At $p_e = 0$, the derivative satisfies

$$f'(0) = 1 + r$$

  - r positive always results in solutions growing away from this equilibrium
  - When the population is small, there are plenty of resources and the population grows (exponentially)
  - Near $p_e = 0$ solutions behave like Malthusian growth

Stability of Logistic Growth Model

Consider the **Carrying Capacity Equilibrium**, $p_e = M$

- Since the **derivative** is

$$f'(p) = 1 + r - \frac{2rp}{M}$$

- At $p_e = M$, the derivative satisfies

$$f'(M) = 1 - r$$

- There are several possible behaviors of the solution near the carrying capacity equilibrium

Behavior of Discrete Dynamical Models

- If $f'(p_e) > 1$
  - Solutions of the discrete dynamical model grow away from the equilibrium (monotonically)
  - **The equilibrium is unstable**
- If $0 < f'(p_e) < 1$
  - Solutions of the discrete dynamical model approach the equilibrium (monotonically)
  - **The equilibrium is stable**
- If $-1 < f'(p_e) < 0$
  - Solutions of the discrete dynamical model oscillate about the equilibrium and approach it
  - **The equilibrium is stable**
- If $f'(p_e) < -1$
  - Solutions of the discrete dynamical model oscillate about the equilibrium but move away from it
  - **The equilibrium is unstable**

Behavior of the Logistic Growth Model

Behavior of Logistic Growth Model near $p_e = M$

- If $0 < r < 1$, then the solution of the discrete logistic model **monotonically approaches the equilibrium**, $p_e = M$, which was observed for the experiment with the yeast
- If $1 < r < 2$, then the solution of the discrete logistic model **oscillates about the equilibrium**, $p_e = M$, but the solution **asymptotically approaches** this equilibrium
- If $2 < r < 3$, then the solution of the discrete logistic model **oscillates about the equilibrium**, $p_e = M$, but the solution **grows away from** this equilibrium
- $r > 3$ results in negative solutions

Example 1 of the Logistic Growth Model 1

Example 1: Consider the **discrete logistic growth model**

$$p_{n+1} = f_1(p_n) = 1.3\,p_n - 0.0001\,p_n^2$$

Skip Example

- Find all the equilibria for this model
- Determine the behavior of the solution near these equilibria
- Sketch a graph of the updating function and the identity map $p_{n+1} = p_n$
- Simulate the model, starting $p_0 = 100$ for 50 iterations

Example 1 of the Logistic Growth Model 2

Solution: For the **discrete logistic growth model**

$$p_{n+1} = 1.3\,p_n - 0.0001\,p_n^2$$

the equilibria are found by substituting $p_e = p_n = p_{n+1}$

Thus,

$$\begin{aligned} p_e &= 1.3\,p_e - 0.0001\,p_e^2 \\ 0 &= 0.3\,p_e - 0.0001\,p_e^2 = p_e(0.3 - 0.0001\,p_e) \end{aligned}$$

The equilibria satisfy

$$p_e = 0$$

and

$$0.3 - 0.0001\,p_e = 0 \quad \text{or} \quad p_e = 3000$$

Example 1 of the Logistic Growth Model 3

Solution (cont): For $f_1(p) = 1.3\,p - 0.0001\,p^2$, the derivative satisfies

$$f_1'(p) = 1.3 - 0.0002\,p$$

At $p_e = 0$

$$f_1'(0) = 1.3 > 1$$

The solution monotonically grows away from this equilibrium, as expected

At $p_e = 3000$

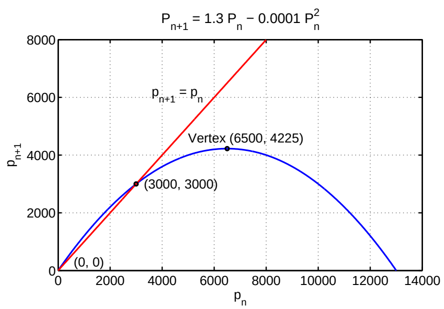
$$f_1'(3000) = 1.3 - 0.6 = 0.7 < 1$$

The solution monotonically approaches this equilibrium
This equilibrium is **stable**

Example 1 of the Logistic Growth Model 4

Graphing the updating function

- The p-intercepts are 0 and 13,000
- The vertex is at (6500, 4225)
- Below is graph of updating function and identity map with significant points



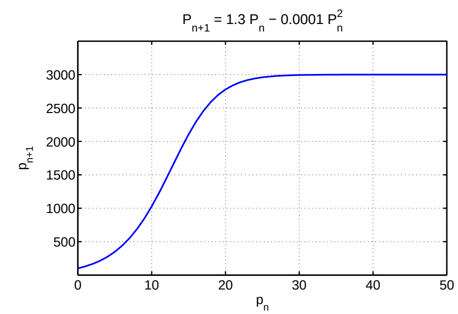
SDSU

Example 1 of the Logistic Growth Model 5

Simulation of

$$p_{n+1} = 1.3\,p_n - 0.0001\,p_n$$

with $p_0 = 100$ for 50 iterations



Shows classic S-curve of population growth

SDSU

Example 2 of the Logistic Growth Model 1

Example 2: Consider the **discrete logistic growth model**

$$p_{n+1} = f_2(p_n) = 2.7\,p_n - 0.0001\,p_n^2$$

Skip Example

- Find all the equilibria for this model
- Determine the behavior of the solution near these equilibria
- Sketch a graph of the updating function and the identity map $p_{n+1} = p_n$
- Simulate the model, starting $p_0 = 100$ for 50 iterations

SDSU

Example 2 of the Logistic Growth Model 2

Solution: For the **discrete logistic growth model**

$$p_{n+1} = 2.7\,p_n - 0.0001\,p_n^2$$

the equilibria are found by substituting $p_e = p_n = p_{n+1}$

Thus,

$$\begin{aligned} p_e &= 2.7\,p_e - 0.0001\,p_e^2 \\ 0 &= 1.7\,p_e - 0.0001\,p_e^2 = p_e(1.7 - 0.0001\,p_e) \end{aligned}$$

The equilibria satisfy

$$p_e = 0$$

and

$$1.7 - 0.0001\,p_e = 0 \quad \text{or} \quad p_e = 17,000$$

SDSU

Example 2 of the Logistic Growth Model 3

Solution (cont): For $f_2(p) = 2.7\,p - 0.0001\,p^2$, the derivative satisfies

$$f_2'(p) = 2.7 - 0.0002\,p$$

At $p_e = 0$

$$f_2'(0) = 2.7 > 1$$

The solution monotonically grows away from this equilibrium, as expected

At $p_e = 17,000$

$$f_2'(17,000) = 2.7 - 3.4 = -0.7$$

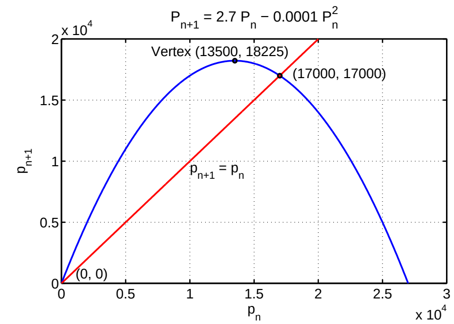
Since $-1 < f_2'(17,000) < 0$, the solution oscillates and approaches this equilibrium

This equilibrium is also **stable**

Example 2 of the Logistic Growth Model 4

Graphing the updating function

- The p-intercepts are 0 and 27,000
- The vertex is at (13500, 18225)
- Below is graph of updating function and identity map with significant points

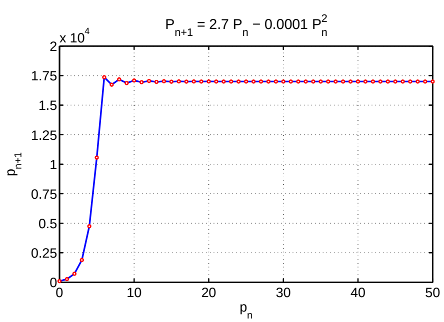


Example 2 of the Logistic Growth Model 5

Simulation of

$$p_{n+1} = 2.7\,p_n - 0.0001\,p_n$$

with $p_0 = 100$ for 50 iterations



Simulation grows and overshoots the equilibrium, then oscillates toward the equilibrium

Example 3 of the Logistic Growth Model 1

Example 3: Consider the **discrete logistic growth model**

$$p_{n+1} = f_3(p_n) = 3.2\,p_n - 0.0001\,p_n^2$$

Skip Example

- Find all the equilibria for this model
- Determine the behavior of the solution near these equilibria
- Sketch a graph of the updating function and the identity map $p_{n+1} = p_n$
- Simulate the model, starting $p_0 = 100$ for 50 iterations

Example 3 of the Logistic Growth Model 2

Solution: For the **discrete logistic growth model**

$$p_{n+1} = 3.2\,p_n - 0.0001\,p_n^2$$

the equilibria are found by substituting $p_e = p_n = p_{n+1}$

Thus,

$$\begin{aligned} p_e &= 3.2\,p_e - 0.0001\,p_e^2 \\ 0 &= 2.2\,p_e - 0.0001\,p_e^2 = p_e(2.2 - 0.0001\,p_e) \end{aligned}$$

The equilibria satisfy

$$p_e = 0$$

and

$$2.2 - 0.0001\,p_e = 0 \quad \text{or} \quad p_e = 22{,}000$$

Example 3 of the Logistic Growth Model 3

Solution (cont): For $f_3(p) = 3.2\,p - 0.0001\,p^2$, the derivative satisfies

$$f_3'(p) = 3.2 - 0.0002\,p$$

At $p_e = 0$

$$f_3'(0) = 3.2 > 1$$

The solution monotonically grows away from this equilibrium, as expected

At $p_e = 22{,}000$

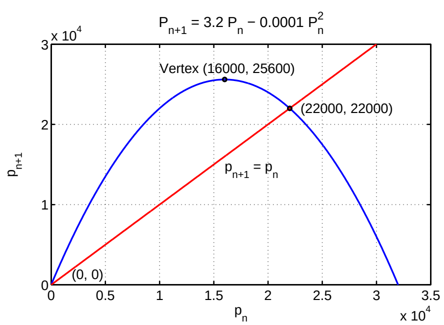
$$f_3'(22{,}000) = 3.2 - 4.4 = -1.2 < -1$$

The solution oscillates away from this equilibrium
This equilibrium is **unstable**

Example 3 of the Logistic Growth Model 4

Graphing the updating function

- The p-intercepts are 0 and 32,000
- The vertex is at (16000, 25600)
- Below is graph of updating function and identity map with significant points

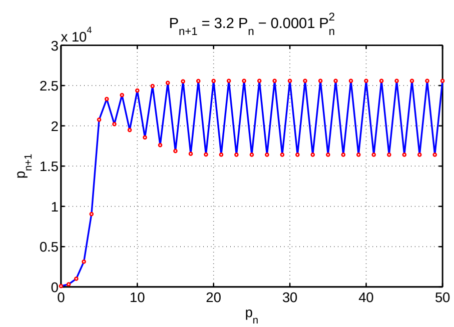


Example 3 of the Logistic Growth Model 5

Simulation of

$$p_{n+1} = 3.2\,p_n - 0.0001\,p_n$$

with $p_0 = 100$ for 50 iterations



Simulation oscillates about the carrying capacity with period 2 behavior

Example 4 - Logistic Growth with Emigration 1

Logistic Growth with Emigration - Population growth may be affected by immigration or emigration

Skip Example

Consider the discrete dynamical population model

$$p_{n+1} = p_n + g(p_n) = 1.71\, p_n - 0.001\, p_n^2 - 7,$$

where n is measured in generations

- This model has a 71% growth rate per generation
- Logistic crowding effects are given by the term $0.001\, p_n^2$
- 7 individuals emigrate each generation

Example 4 - Logistic Growth with Emigration 2

Logistic Growth with Emigration

$$p_{n+1} = p_n + g(p_n) = 1.71\, p_n - 0.001\, p_n^2 - 7,$$

- Let $p_0 = 100$ and find the population for the next 3 generations
- Find the p-intercepts and the vertex for $g(p)$ and graph of $g(p)$
- By finding when the growth rate is zero, determine all equilibria for this model and analyze their stability

Example 4 - Logistic Growth with Emigration 3

Solution: We begin with $p_0 = 100$

$$\begin{aligned} p_1 &= p_0 + g(p_0) = 100 + 0.71(100) - 0.001(100)^2 - 7 = 154, \\ p_2 &= 154 + 0.71(154) - 0.001(154)^2 - 7 = 233, \\ p_3 &= 233 + 0.71(233) - 0.001(233)^2 - 7 = 337. \end{aligned}$$

Example 4 - Logistic Growth with Emigration 4

Solution (cont): The growth function satisfies

$$\begin{aligned} g(p) &= 0.71p - 0.001p^2 - 7 \\ g(p) &= -0.001(p^2 - 710p + 7000) \\ g(p) &= -0.001(p - 10)(p - 700) \end{aligned}$$

The p-intercepts are

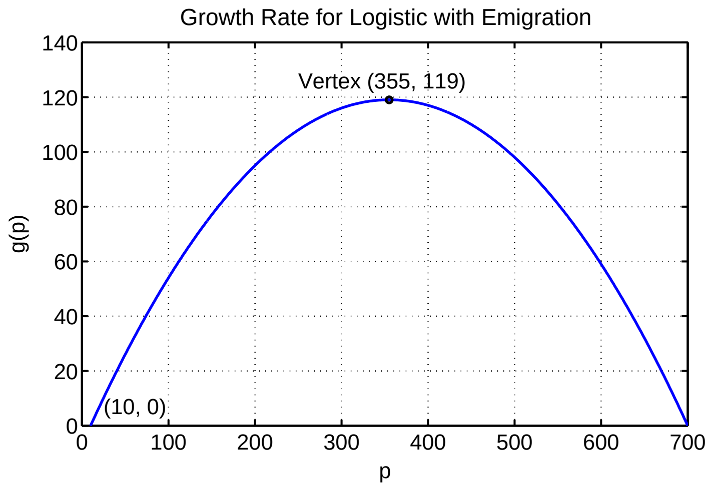
$$p = 10 \qquad \text{or} \qquad p = 700$$

The vertex satisfies $p = 355$ with

$$g(355) = -0.001(345)(-345) = 119$$

Example 4 - Logistic Growth with Emigration 5

Solution (cont): The graph of the growth function is



Example 4 - Logistic Growth with Emigration 6

Solution (cont): Equilibrium Analysis
Since the growth function $g(p)$ is zero at

$$p = 10 \qquad \text{and} \qquad p = 700,$$

these are the **equilibria**

The updating function is

$$F(p) = 1.71\,p - 0.001\,p^2 - 7$$

with derivative

$$F'(p) = 1.71 - 0.002\,p$$

Example 4 - Logistic Growth with Emigration 7

Solution (cont): Stability Analysis With

$$F'(p) = 1.71 - 0.002\,p$$

At $p = 10$,

$$F'(10) = 1.69 > 1$$

so this equilibrium is **monotonically unstable (solutions growing away)**

At $p = 700$,

$$F'(700) = 0.31 < 1$$

so this equilibrium is **monotonically stable (solutions moving toward)**

Example 5 - U. S. Census with 3 Growth Models 1

U. S. Census with Logistic Growth Model - This example uses the census data from 1790 to 2000 to compare 3 models

Skip Example

- Malthusian growth model

$$P_{n+1} = 1.1524\,P_n$$

- Nonautonomous growth model with n in decades after 1790

$$P_{n+1} = (1.3768 - 0.01473\,n)P_n$$

- Logistic growth model

$$P_{n+1} = f(P_n) = P_n + 0.2334\,P_n\left(1 - \frac{P_n}{411.1}\right)$$

Example 5 - U. S. Census with 3 Growth Models 2

Malthusian growth model

$$P_{n+1} = (1+r)P_n \quad \text{with} \quad P_0$$

- Least squares best fit to census data

$$P_n = P_0(1+r)^n = 15.05(1.1524)^n$$

- The average growth over U. S. census history is $r = 0.1524$ per decade with best $P_0 = 15.05$ M
  - The sum of square errors is 2248
  - The P_0 is quite high and growth only matches growth near beginning of 20^{th} century
- Malthusian model isn't expected to work well over long periods of time

SDSU

Example 5 - U. S. Census with 3 Growth Models 3

Nonautonomous growth model

$$P_{n+1} = (1+k(t_n))P_n \quad \text{with} \quad P_0$$

- Best linear fit to growth over U. S. history is

$$k(t_n) = 0.3768 - 0.01473\, n$$

- Growth near 38% per decade early, declining about 1.5% per decade
- Least squares best fit to census data had $P_0 = 3.77$ M
  - The sum of square errors is 543
  - The P_0 is very close to actual 1790 census
- This model matches the census quite well, but model difficult to analyze mathematically

SDSU

Example 5 - U. S. Census with 3 Growth Models 4

Logistic growth model

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{M}\right) \quad \text{with} \quad P_0$$

- Least squares best fit to census data

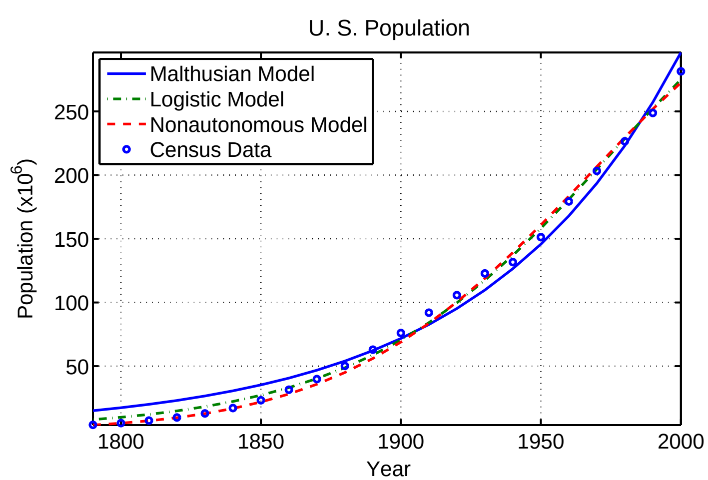
$$P_{n+1} = P_n + 0.2334\, P_n\left(1 - \frac{P_n}{411.1}\right)$$

- This gives a growth rate of $r = 0.2334$ and carrying capacity of $M = 411.1$ with the best $P_0 = 8.04$
  - The sum of square errors is 479
  - The P_0 is high at 8.04 M
- This model matches the census data best of the 3 models

SDSU

Example 5 - U. S. Census with 3 Growth Models 5

Graph of the 3 models and U. S. census data



SDSU

Example 5 - U. S. Census with 3 Growth Models

Logistic Updating Function

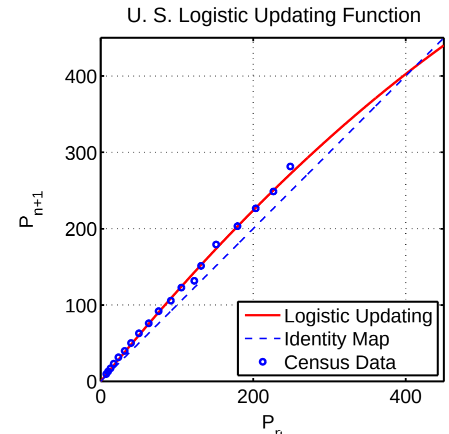
- Direct fitting of the logistic time series to data can be numerically unstable
- Finding the quadratic updating function uses stable numerical routines
- By plotting P_{n+1} versus P_n, one can see how the data compares to the updating function for the logistic growth model
- Find P_n and P_{n+1} by taking successive pairs of census data

Discrete Logistic Growth Model
Qualitative Analysis of Logistic Growth Model
Cobwebbing
Equilibria
Simulation of Logistic Growth Model
Stability of Logistic Growth Model
Behavior of Discrete Dynamical Models
Examples of Logistic Growth
U. S. Population Models

Example 5 - U. S. Census with 3 Growth Models

Graph of the Logistic Updating function

Graph shows U. S. census data, quadratic for logistic model, and identity map



Discrete Logistic Growth Model
Qualitative Analysis of Logistic Growth Model
Cobwebbing
Equilibria
Simulation of Logistic Growth Model
Stability of Logistic Growth Model
Behavior of Discrete Dynamical Models
Examples of Logistic Growth
U. S. Population Models

Example 5 - U. S. Census with 3 Growth Models

Logistic Updating function for U. S. census data

- The logistic updating function very closely follows the census data except at a couple of points
- The equilibria occur at the intersection of the updating function and the identity map
- The slope of the updating function at a point of intersection determines the stability of that equilibrium

Discrete Logistic Growth Model
Qualitative Analysis of Logistic Growth Model
Cobwebbing
Equilibria
Simulation of Logistic Growth Model
Stability of Logistic Growth Model
Behavior of Discrete Dynamical Models
Examples of Logistic Growth
U. S. Population Models

Example 5 - U. S. Census with 3 Growth Models

Logistic Updating function for U. S. census data

$$f(P_n) = P_n + 0.2334\,P_n\left(1 - \frac{P_n}{411.1}\right) = 1.2334\,P_n - 0.00056775\,P_n^2$$

The **equilibria** satisfy

$$\begin{aligned} P_e &= P_e + 0.2334\,P_e\left(1 - \frac{P_e}{411.1}\right) \\ 0 &= 0.2334\,P_e\left(1 - \frac{P_e}{411.1}\right) \end{aligned}$$

The equilibria are

$$P_e = 0 \qquad \text{and} \qquad P_e = 411.1$$

Example 5 - U. S. Census with 3 Growth Models 10

Updating Function

$$f(P) = 1.2334\,P - 0.00056775\,P^2$$

The **derivative of the updating function** is

$$f'(P) = 1.2334 - 0.0011355\,P$$

At the equilibrium, $P_e = 0$,

$$f'(0) = 1.2334 > 1$$

This equilibrium is **unstable** with solutions monotonically moving away

Example 5 - U. S. Census with 3 Growth Models 11

Since the **derivative of the updating function** is

$$f'(P) = 1.2334 - 0.0011355\,P$$

At the equilibrium, $P_e = 411.1$,

$$f'(411.1) = 0.7666 < 1$$

This equilibrium is **stable** with solutions monotonically approaching the **carrying capacity**

Example 5 - U. S. Census with 3 Growth Models 12

Summary: Future Projections

- The Malthusian growth model is simple but simulates poorly for the entire history of the U. S.
- Nonautonomous growth model
  - Simulates historical data well, but low by 3.2% in 2000 and 5.8% in 2010
  - Fails to account for recent immigration and high birth rates in immigrant community
  - Model predicts population increases to a maximum of 330 M around 2050, then declines
- Logistic growth model
  - Simulates historical data well, but low by 2.3% in 2000 and 4.0% in 2010 missing importance of recent immigration
  - Model predicts population increases to carrying capacity of 411.1 M, asymptotically

Cobwebbing 1

Consider the **discrete dynamical model**

$$p_{n+1} = f(p_n)$$

In the **Linear Discrete Dynamical Model section**, we showed a graphical method to view the local dynamics of this model called **cobwebbing**

Create a graph with the variable p_{n+1} on the vertical axis and p_n on the horizontal axis

Draw the graph of the **updating function**, $f(p_n)$ and the **identity map**

$$p_{n+1} = f(p_n) \qquad \text{and} \qquad p_{n+1} = p_n$$

Cobwebbing 2

Graphically, any intersection of the **updating function** and the **identity map**

$$p_{n+1} = f(p_n) \qquad \text{and} \qquad p_{n+1} = p_n$$

produces an equilibrium

- The process of **cobwebbing** shows the dynamics of this discrete dynamical model
- Start at some point p_0 on the horizontal axis, then go vertically to $f(p_0)$ to find p_1
- Next go horizontally to the line $p_{n+1} = p_n$
- Go vertically to $f(p_1)$ to find p_2
- The process is repeated to give a geometric interpretation of the dynamics of the discrete model

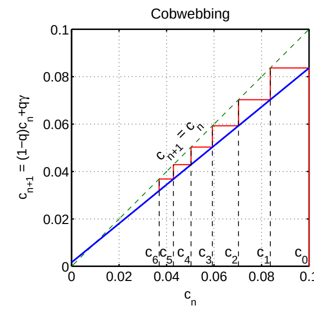
Cobwebbing – Breathing Model Example

Cobwebbing – Breathing Model Example

The model for a normal subject breathing an air mixture enriched with Ar satisfies the model

$$c_{n+1} = (1-q)c_n + q\gamma = 0.82\,c_n + 0.0017$$

Below reviews the cobwebbing process for this example



Cobwebbing – Quadratic Example 1

Cobwebbing – Quadratic Example

- Breathing model has a simple **linear updating function**
  - Unique equilibrium
  - Monotonic dynamics
- **Quadratic updating function** allows complicated dynamics
  - Logistic growth model is a quadratic dynamical model
  - Have observed monotonic, oscillatory, and chaotic dynamics
  - Show oscillatory dynamics for

    $$p_{n+1} = 3\,p_n(1-p_n)$$

    using a few steps of cobwebbing
  - This example has equilibria at 0 and $\frac{2}{3}$, the latter being between stable and unstable and oscillatory

Cobwebbing – Quadratic Example 2

Cobwebbing – Quadratic Example