Calculus for the Life Sciences II

Lecture Notes – Linear Differential Equations

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Introduction
Blood Pressure
Radioactive Decay
Solution of Linear Growth and Decay Models
Newton's Law of Cooling
Solution of General Linear Model
Pollution in a Lake

Outline

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Introduction

Introduction

- Examples of linear first order differential equations
  - Arterial blood pressure
  - Radioactive decay
  - Newton's law of cooling
  - Pollution in a Lake
- Extend earlier techniques to find solutions

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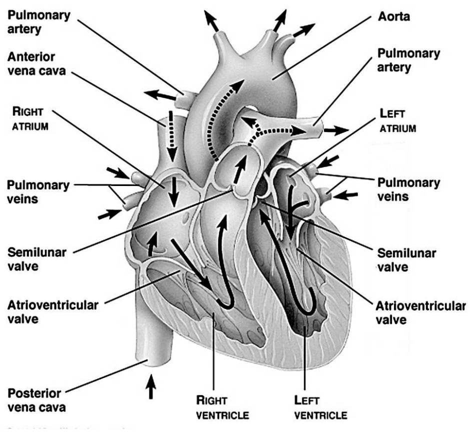
Blood Pressure

Blood Pressure

- **Blood Pressure** is divided into **systolic** and **diastolic** pressure
- Normal reading is **120/80** (in mm of Hg)
- How are those numbers generated and what can we infer from them?
- The numbers for blood pressure reflect the force on arterial walls
- This pressure is generated by the beating of the heart

Blood Pressure

Diagram of Heart



Cardiac Cycle 1

Cardiac Cycle

- Pulmonary circulation
  - Blood flows from the body into the right atrium
  - Flows to the right ventricle
  - Blood goes through the pulmonary artery to the lungs
  - Blood exchanges O_2 and CO_2 in the lungs
  - Blood returns through the pulmonary vein to the left atrium
  - Pressure in the pulmonary vein and left atrium is between 5 and 15 mm of Hg
- Blood flows into the left ventricle
- The heart is rigid, so pressure increases only slightly
- The right atrium contracts, then the AV valve between the atrium and the ventricle closes

Cardiac Cycle 2

Cardiac Cycle (cont)

- The heart receives an electrical signal, which causes ventricular contraction, beginning **systole**
- The left ventricle contracts, and the pressure increases until it "blows" open the aortic valve
- Blood rapidly flows into the aorta under this high pressure (**systolic pressure**)
- As pressure rises in the aorta, the AV valve reopens, while the aortic valve closes
- Now there is high pressure in the aorta, which forces the blood into the other arteries of the body
- As the blood flows through the body, the aortic pressure drops to its low pressure, the **diastolic pressure**

Arterial Blood Pressure 1

Arterial Blood Pressure: Model the arterial pressure, $P_a(t)$, during a single beat of the heart

- Determine the important modeling parameters in the system
- The cardiac output, Q, represents the average amount of blood pumped by the heart (in liters/min)
- The stroke volume, V, is the amount of blood pumped by the heart during one beat (liters/beat)
- T is the duration of a heart beat
- Relate flow from the cardiac output to the stroke volume by the relationship

$$\text{Cardiac Output} = \text{Stroke Volume} \;/\; \text{Duration of the Flow}$$

$$Q = V/T \quad \text{(liters/min)}$$

Arterial Blood Pressure 2

Arterial Blood Pressure:

- The left ventricle completes pumping the blood into the aorta and the aortic valve closes at the maximum pressure, P_{sys}
- The blood pressure begins to fall as the blood flows through the arteries
- The rate of flowing of the blood depends on the **resistance** of a blood vessel
  - Viscosity of the blood
  - Length of the vessels
  - Radius of the blood vessels

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Arterial Blood Pressure 3

Arterial Blood Pressure:

- The viscosity of the blood is relatively constant, except under diseased states like erythrocytemia (or when athletes take erythropoietin or EPO to overstimulate the production of red blood cells)
- The length of the blood vessels are relatively constant, except for when conditions like pregnancy or amputation occur
- The main factor that changes resistance of the blood flow is change in the radius
- Blood pressure becomes a valuable tool for detecting narrowing of the blood vessels by hypertension or atherosclerosis

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Modeling Blood Pressure 1

Modeling Blood Pressure:

- Experimentally, it has been observed that **systemic blood flow**, Q_s, is proportional to the difference between the arterial and venous pressures $(P_a(t) - P_v(t))$ with the proportionality dependent on the resistance
- If R_s is the systemic resistance (mm Hg/liter/min), then we have the following equation:

$$Q_s(t) = \frac{1}{R_s}(P_a(t) - P_v(t))$$

- To simplify the model, we take advantage of the fact that venous pressures are very low, so we approximate the systemic flow by the equation:

$$Q_s(t) = \frac{1}{R_s} P_a(t)$$

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Modeling Blood Pressure 2

Compliance:

- **Compliance** is the stretchability of a vessel, which is a property that allows a vessel to change the volume in response to pressure changes
- The higher the compliance the easier it is for a vessel to expand in response to increased pressure
- Resistance and compliance have a roughly inverse relationship
- Experimentally, the arterial volume, Va, is roughly equal to the compliance, C_a, times the arterial pressure

$$V_a(t) = C_a P_a(t)$$

Modeling Blood Pressure 3

Differential Equation for Blood Flow:

- The flow representing the change in the arterial volume is given by the difference between the rate of flow entering the aorta and the rate of flow from the aorta
- Since the aortic valve is closed during systole, no blood is entering the aorta
- The differential equation is

$$\frac{dV_a(t)}{dt} = \text{flow rate in} \; - \; \text{flow rate out} = 0 - Q_s(t)$$

- Thus,

$$\frac{dV_a(t)}{dt} = -\frac{1}{R_s}P_a(t)$$

Modeling Blood Pressure 4

Differential Equation for Blood Flow: Since $V_a(t) = C_a P_a(t)$,

$$\frac{dV_a(t)}{dt} = C_a \frac{dP_a(t)}{dt}$$

This gives the **initial value problem**

$$\frac{dP_a(t)}{dt} = -\frac{1}{C_a R_s}P_a(t) \quad \text{with} \quad P_a(0) = P_{sys}$$

The **solution** is

$$P_a(t) = P_{sys}e^{-\frac{t}{C_a R_s}} \quad \text{for} \quad t \in [0, T]$$

Diagnosis with Model 1

Diagnosis with Model: How can this model be used to provide a non-invasive method for estimating the physiological parameters for compliance, C_a , and resistance, R_s

- Measurable physiological quantities are
  - The heart rate or pulse, $\frac{1}{T}$
  - Cardiac output, Q, using a doppler sonogram
  - The systolic and diastolic pressures, P_{sys} and P_{dia}
- **Compliance** comes from the stroke volume, V,

$$V = V_{sys} - V_{dia} = C_a P_{sys} - C_a P_{dia}$$

- But $V = QT$, so **compliance** satisfies

$$C_a = \frac{QT}{P_{sys} - P_{dia}}$$

Diagnosis with Model 2

Resistance: The model gives the diastolic pressure just before the next heart beat

$$P_{dia} = P_{sys}e^{-\frac{T}{C_a R_s}}$$

Solve this equation for the **resistance**, R_s

$$R_s = \frac{T}{C_a\left(\ln(P_{sys}) - \ln(P_{dia})\right)}$$

- Normal Person
  - Pulse of approximately 70 beats/min ($\frac{1}{T}$)
  - Cardiac output of $Q = 5.6$ (liters/min)
  - Systolic and diastolic pressures of $P_{sys} = 120$ mm Hg and $P_{dia} = 80$ mm Hg
- Compute the compliance and resistance for a normal person

$C_a = 0.002$ (liters/mm Hg) and $R_s = 17.6$ (mm Hg/liter/min)

Example of Athlete 1

Example of an Athlete: Consider a trained athlete considered in very good condition

- Suppose an athlete has
  - A pulse of **60 beats/min (at rest)**
  - A blood pressure of **120/75**
  - A measured cardiac output of **6 liters/min**
- Find the **compliance,** C_a, and systemic **resistance,** R_s, of the arteries for this individual

Example of Athlete 2

Solution: From the formula, **compliance,** C_a

$$C_a = \frac{QT}{P_{sys} - P_{dia}} = \frac{6.0/60}{120-75} = 0.00222 \text{ (liters/mm Hg)}$$

This is slightly larger than for a normal person

The **systemic resistance,** R_s, satisfies

$$\begin{aligned} R_s &= \frac{T}{C_a\,(\ln(P_{sys}) - \ln(P_{dia}))} \\ &= \frac{1/60}{0.00222(\ln(120) - \ln(75))} = 15.96 \text{ (mm Hg/liter/min)} \end{aligned}$$

This is lower than for a normal person, which is what we would expect for someone in better condition

Radioactive Decay

Radioactive Decay: Radioactive elements are important in many biological applications

- ^{3}H (tritium) is used to tag certain DNA base pairs
  - Add to mutant strains of *E. coli* that are unable to manufacture one particular DNA base
  - Using antibiotics, one uses the radioactive signal to determine how much DNA is replicated under a particular set of experimental conditions
- Radioactive iodine is often used to detect or treat thyroid problems
- Most experiments are run so that radioactive decay is not an issue
  - ^{3}H has a half-life of 12.5 yrs
  - ^{131}I has a half-life of 8 days

Carbon Radiodating 1

Carbon Radiodating: One important application of radioactive decay is the dating of biological specimens

- A living organism is continually changing its carbon with the environment
  - Plants directly absorb CO_2 from the atmosphere
  - Animals get their carbon either directly or indirectly from plants
- Gamma radiation that bombards the Earth keeps the ratio of ^{14}C to ^{12}C fairly constant in the atmospheric CO_2
- ^{14}C stays at a constant concentration until the organism dies

Carbon Radiodating 2

Modeling Carbon Radiodating: Radioactive carbon, ^{14}C, decays with a **half-life of 5730 yr**

- Living tissue shows a radioactivity of about 15.3 disintegrations per minute (dpm) per gram of carbon
- The loss of ^{14}C from a sample at any time t is proportional to the amount of ^{14}C remaining
- Let $R(t)$ be the dpm per gram of ^{14}C from an ancient object
- The differential equation for a gram of ^{14}C

$$\frac{dR(t)}{dt} = -kR(t) \qquad \text{with} \qquad R(0) = 15.3$$

- This differential equation has the solution

$$R(t) = 15.3\, e^{-kt}, \quad \text{where} \quad k = \frac{\ln(2)}{5730} = 0.000121$$

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Example: Carbon Radiodating

Example Carbon Radiodating: Suppose that an object is found to have a radioactive count of 5.2 dpm per g of carbon

Find the age of this object

Solution: From above

$$\begin{aligned} 5.2 &= 15.3\, e^{-kt} \\ e^{kt} &= \tfrac{15.3}{5.2} = 2.94 \\ kt &= \ln(2.94) \end{aligned}$$

Thus, $t = \frac{\ln(2.94)}{k} = 8915$ yr, so the object is about 9000 yrs old

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Hyperthyroidism 1

Hyperthyroidism is a serious health problem caused by an overactive thyroid

- The primary hormone released is **thyroxine**, which stimulates the release of other hormones
- Too many other hormones, such as insulin and the sex hormones
- Result is low blood sugar causing lethargy or mood disorders and sexual dysfunction
- One treatment for hyperthyroidism is **ablating the thyroid** with a large dose of **radioactive iodine, ^{131}I**
  - The thyroid concentrates iodine brought into the body
  - The ^{131}I undergoes both β and γ radioactive decay, which destroys tissue
  - Patient is given medicine to supplement the loss of thyroxine

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Hyperthyroidism 2

Hyperthyroidism: Treatment

- Based upon the thyroid condition and body mass, a standard dose ranges from **110-150 mCi** (milliCuries), given in a special "cocktail"
- It is assumed that almost 100% of the ^{131}I is absorbed by the blood from the gut
- The thyroid uptakes 30% of this isotope of iodine, peaking around 3 days
- The remainder is excreted in the urine
- The half-life of ^{131}I is 8 days, so this isotope rapidly decays
- Still the patient must remain in a designated room for 3-4 days for this procedure, so that he or she does not irradiate the public from his or her treatment

Hyperthyroidism

Hyperthyroidism Example: Assume that a patient is given a **120 mCi** cocktail of ^{131}**I** and that **30%** is absorbed by the thyroid

- Find the amount of ^{131}**I** in the thyroid (in mCi), if the patient is released four days after swallowing the radioactive cocktail
- Calculate how many mCis the patient's thyroid retains after 30 days, assuming that it was taken up by the thyroid and not excreted in the urine

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Hyperthyroidism

Solution:

- Assume for simplicity of the model that the ^{131}**I** is immediately absorbed into the thyroid, then stays there until it undergoes radioactive decay
- Since the thyroid uptakes **30%** of the **120 mCi**, assume that the thyroid has **36 mCi** immediately after the procedure
- This is an oversimplification as it takes time for the ^{131}**I** to accumulate in the thyroid
- This allows the simple model

$$\frac{dR}{dt} = -k\,R(t) \qquad \text{with} \qquad R(0) = 36 \text{ mCi}$$

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Solution (cont): The radioactive decay model is

$$\frac{dR}{dt} = -k\,R(t) \qquad \text{with} \qquad R(0) = 36 \text{ mCi}$$

- The solution is
$$R(t) = 36\,e^{-kt}$$
- Since the half-life of ^{131}**I** is 8 days, after 8 days there will are 18 mCi of ^{131}**I**
- Thus, $R(8) = 18 = 36\,e^{-8k}$, so
$$e^{8k} = 2 \qquad \text{or} \qquad 8k = \ln(2)$$
- Thus, $k = \frac{\ln(2)}{8} = 0.0866 \text{ day}^{-1}$

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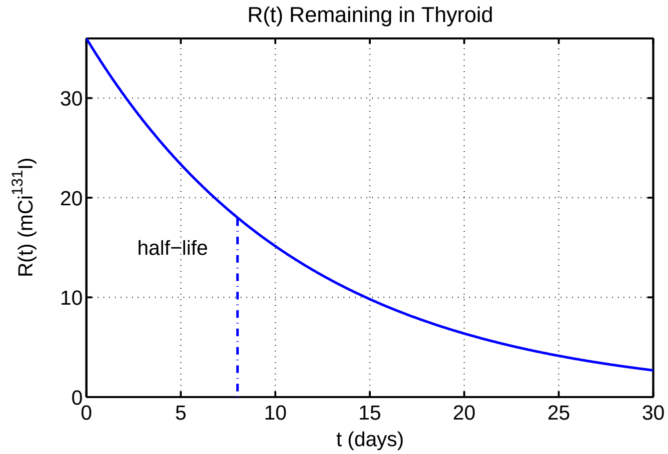
Solution (cont): Since

$$R(t) = 36\,e^{-kt} \qquad \text{with} \qquad k = 0.0866 \text{ day}^{-1}$$

- At the time of the patient's release t = 4 days, so in the thyroid
$$R(4) = 36\,e^{-4k} = \frac{36}{\sqrt{2}} = 25.46 \text{ mCi}$$
- After 30 days, we find in the thyroid
$$R(30) = 36\,e^{-30k} = 2.68 \text{ mCi}$$

Hyperthyroidism

Graph of $R(t)$



Solution of Linear Growth and Decay Models

General Solution to Linear Growth and Decay Models: Consider

$$\frac{dy}{dt} = a\,y \qquad \text{with} \qquad y(t_0) = y_0$$

The solution is

$$y(t) = y_0 e^{a(t-t_0)}$$

Example: Linear Decay Model

Example: Linear Decay Model: Consider

$$\frac{dy}{dt} = -0.3\,y \qquad \text{with} \qquad y(4) = 12$$

The solution is

$$y(t) = 12\,e^{-0.3(t-4)}$$

Newton's Law of Cooling

Newton's Law of Cooling:

- After a murder (or death by other causes), the forensic scientist takes the temperature of the body
- Later the temperature of the body is taken again to find the rate at which the body is cooling
- Two (or more) data points are used to extrapolate back to when the murder occurred
- This property is known as **Newton's Law of Cooling**

Newton's Law of Cooling 2

Newton's Law of Cooling states that the rate of change in temperature of a cooling body is proportional to the difference between the temperature of the body and the surrounding environmental temperature

- If $T(t)$ is the temperature of the body, then it satisfies the differential equation

$$\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T(0) = T_0$$

- The parameter k is dependent on the specific properties of the particular object (body in this case)
- T_e is the environmental temperature
- T_0 is the initial temperature of the object

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Murder Example 1

Murder Example

- Suppose that a murder victim is found at 8:30 am
- The temperature of the body at that time is 30°C
- Assume that the room in which the murder victim lay was a constant 22°C
- Suppose that an hour later the temperature of the body is 28°C
- Normal temperature of a human body when it is alive is 37°C
- Use this information to determine the approximate time that the murder occurred

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Murder Example 2

Solution: From the model for Newton's Law of Cooling and the information that is given, if we set $t = 0$ to be 8:30 am, then we solve the initial value problem

$$\frac{dT}{dt} = -k(T(t) - 22) \quad \text{with} \quad T(0) = 30$$

- Make a change of variables $z(t) = T(t) - 22$
- Then $z'(t) = T'(t)$, so the differential equation above becomes

$$\frac{dz}{dt} = -kz(t), \quad \text{with} \quad z(0) = T(0) - 22 = 8$$

- This is the radioactive decay problem that we solved
- The solution is

$$z(t) = 8\,e^{-kt}$$

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Murder Example 3

Solution (cont): From the solution $z(t) = 8\,e^{-kt}$, we have

$$\begin{aligned} z(t) &= T(t) - 22, \quad \text{so} \quad T(t) = z(t) + 22 \\ T(t) &= 22 + 8\,e^{-kt} \end{aligned}$$

- One hour later the body temperature is 28°C

$$T(1) = 28 = 22 + 8\,e^{-k}$$

- Solving

$$6 = 8\,e^{-k} \quad \text{or} \quad e^{k} = \tfrac{4}{3}$$

- Thus, $k = \ln\left(\frac{4}{3}\right) = 0.2877$

Murder Example 4

Solution (cont): It only remains to find out when the murder occurred

- At the time of death, t_d, the body temperature is 37°C

$$T(t_d) = 37 = 22 + 8\,e^{-k}$$

- Thus,

$$8\,e^{-kt_d} = 37 - 22 = 15 \quad \text{or} \quad e^{-kt_d} = \frac{15}{8} = 1.875$$

- This gives $-kt_d = \ln(1.875)$ or

$$t_d = -\frac{\ln(1.875)}{k} = -2.19$$

- The murder occurred about 2 hours 11 minutes before the body was found, which places the time of death around **6:19 am**

Cooling Tea 1

Cooling Tea: We would like to determine whether a cup of tea cools more rapidly by adding cold milk right after brewing the tea or if you wait 5 minutes to add the milk

- Begin with $\frac{4}{5}$ cup of boiling hot tea, $T(0) = 100$°C
- Assume the tea cools according to Newton's law of cooling

$$\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T_e = 20°\text{C}$$

- k is the cooling constant based on the properties of the cup to be calculated
- a. In the first scenario, you let the tea cool for 5 minutes, then add $\frac{1}{5}$ cup of cold milk, 5°C

Cooling Tea 2

Cooling Tea (cont):

- Assume that after 2 minutes the tea has cooled to a temperature of 95°C
- Determine the value of k, which we assume stays the same in this problem
- Mix in the milk, assuming that the temperature mixes perfectly in proportion to the volume of the two liquids
- b. In the second case, add $\frac{1}{5}$ cup of cold milk, 5°C, immediately and mix it thoroughly
- **Find how long until each cup of tea reaches a temperature of 70°C**

Cooling Tea 3

Solution of Cooling Tea: Find the rate constant k for

$$\frac{dT}{dt} = -k(T(t) - 20), \qquad T(0) = 100 \quad \text{and} \quad T(2) = 95$$

- Let $z(t) = T(t) - 20$, so $z(0) - T(0) - 20 = 80$
- Since $z'(t) = T'(t)$, the initial value problem becomes

$$\frac{dz}{dt} = -k\,z(t), \qquad z(0) = 80$$

- The solution is

$$z(t) = 80\,e^{-kt} = T(t) - 20$$

- Thus,

$$T(t) = 80\,e^{-kt} + 20$$

Cooling Tea 4

Solution (cont): The solution is

$$T(t) = 80\,e^{-kt} + 20$$

- Since $T(2) = 95$,
$$95 = 80e^{-2k} + 20 \qquad \text{or} \qquad e^{2k} = \frac{80}{75}$$
  - $k = \frac{\ln\left(\frac{80}{75}\right)}{2} = 0.03227$
- Find the temperature at 5 min
$$T(5) = 80e^{-5k} + 20 = 88.1^\circ\text{C}$$
- Now mix the $\frac{4}{5}$ cup of tea at 88.1°C with the $\frac{1}{5}$ cup of milk at 5°C, so
$$T_+(5) = 88.1\left(\tfrac{4}{5}\right) + \left(5\tfrac{1}{5}\right) = 71.5^\circ\text{C}$$

Cooling Tea 5

Solution (cont): For the first scenario, the temperature after adding the milk after 5 min satisfies

$$T_+(5) = 71.5^\circ\text{C}$$

- The new initial value problem is
$$\frac{dT}{dt} = -k(T(t) - 20), \qquad T(5) = 71.5^\circ\text{C}$$
- With the same substitution, $z(t) = T(t) - 20$,
$$\frac{dz}{dt} = -kz, \qquad z(5) = 51.5$$
- This has the solution
$$z(t) = 51.5e^{-k(t-5)} = T(t) - 20$$

Cooling Tea 6

Solution (cont): For the first scenario, the temperature satisfies

$$T(t) = 51.5e^{-k(t-5)} + 20$$

- To find when the tea is 70°C, solve
$$70 = 51.5e^{-k(t-5)} + 20$$
- Thus,
$$e^{k(t-5)} = \frac{51.5}{50}$$
- It follows that $k(t-5) = \ln(51.5/50)$, so
$$t = 5 + \frac{\ln(51.5/50)}{k} = 5.92 \text{ min}$$

Cooling Tea 7

Solution (cont): For the second scenario, we mix the tea and milk, so

$$T(0) = 100\left(\tfrac{4}{5}\right) + 5\left(\tfrac{1}{5}\right) = 81^\circ\text{C}$$

- The new initial value problem is
$$\frac{dT}{dt} = -k(T(t) - 20), \qquad T(0) = 81^\circ\text{C}$$
- With $z(t) = T(t) - 20$,
$$\frac{dz}{dt} = -k\,z(t), \qquad z(0) = 61$$
- This has the solution
$$z(t) = 61e^{-kt} = T(t) - 20$$

Cooling Tea 8

Solution (cont): For the second scenario, the solution is

$$T(t) = 61\,e^{-kt} + 20$$

- To find when the tea is 70°C, solve

$$70 = 61e{-}kt + 20$$

- Thus,

$$e^{kt} = \frac{61}{50}$$

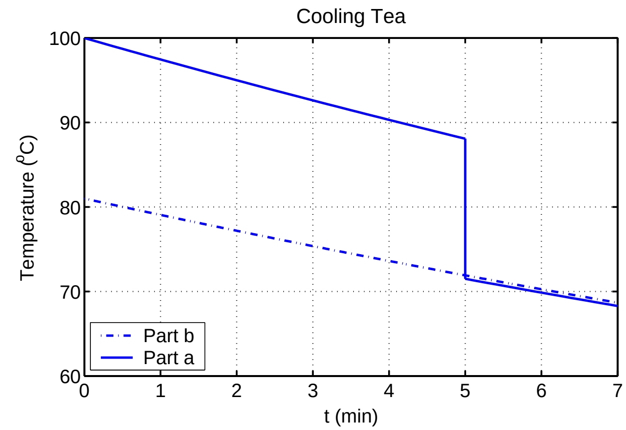
- Since $kt = \ln\left(\frac{61}{50}\right)$,

$$t = \frac{\ln(61/50)}{k} = 6.16 \text{ min}$$

- Waiting to pour in the milk for 5 minutes, saves about 15 seconds in cooling time

Newton's Law of Cooling 9

Graph of Cooling Tea



Solution of General Linear Model 1

Solution of General Linear Model Consider the Linear Model

$$\frac{dy}{dt} = a\,y + b \qquad \text{with} \qquad y(t_0) = y_0$$

Rewrite equation as

$$\frac{dy}{dt} = a\left(y + \frac{b}{a}\right)$$

Make the substitution $z(t) = y(t) + \frac{b}{a}$, so $\frac{dz}{dt} = \frac{dy}{dt}$ and $z(t_0) = y_0 + \frac{b}{a}$

$$\frac{dz}{dt} = a\,z \qquad \text{with} \qquad z(t_0) = y_0 + \frac{b}{a}$$

Solution of General Linear Model 2

Solution of General Linear Model The shifted model is

$$\frac{dz}{dt} = a\,z \qquad \text{with} \qquad z(t_0) = y_0 + \frac{b}{a}$$

The solution to this problem is

$$z(t) = \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} = y(t) + \frac{b}{a}$$

The solution is

$$y(t) = \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a}$$

Example of Linear Model 1

Example of Linear Model Consider the Linear Model

$$\frac{dy}{dt} = 5 - 0.2\,y \qquad \text{with} \qquad y(3) = 7$$

Rewrite equation as

$$\frac{dy}{dt} = -0.2(y - 25)$$

Make the substitution $z(t) = y(t) - 25$, so $\frac{dz}{dt} = \frac{dy}{dt}$ and $z(3) = -18$

$$\frac{dz}{dt} = -0.2\,z \qquad \text{with} \qquad z(3) = -18$$

Example of Linear Model 2

Example of Linear Model The substituted model is

$$\frac{dz}{dt} = -0.2\,z \qquad \text{with} \qquad z(3) = -18$$

Thus,

$$z(t) = -18\,e^{-0.2(t-3)} = y(t) - 25$$

The solution is

$$y(t) = 25 - 18\,e^{-0.2(t-3)}$$

Pollution in a Lake 1

Pollution in a Lake: Introduction

- One of the most urgent problems in modern society is how to reduce the pollution and toxicity of our water sources
- These are very complex issues that require a multidisciplinary approach and are often politically very intractable because of the key role that water plays in human society and the many competing interests
- Here we examine a very simplistic model for pollution of a lake
- The model illustrates some basic elements from which more complicated models can be built and analyzed

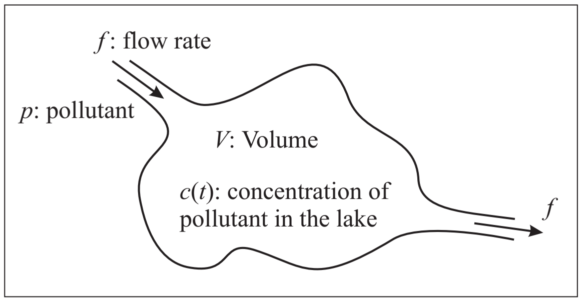
Pollution in a Lake 2

Pollution in a Lake: Problem set up

- Consider the scenario of a new pesticide that is applied to fields upstream from a clean lake with volume V
- Assume that a river receives a constant amount of this new pesticide into its water, and that it flows into the lake at a constant rate, f
- This assumption implies that the river has a constant concentration of the new pesticide, p
- Assume that the lake is well-mixed and maintains a constant volume by having a river exiting the lake with the same flow rate, f, of the inflowing river

Newton's Law of Cooling

Diagram for Lake Problem Design a model using a linear first order differential equation for the concentration of the pesticide in the lake, $c(t)$



Pollution in a Lake

Differential Equation for Pollution in a Lake

- Set up a differential equation that describes the mass balance of the pollutant
- **The change in amount of pollutant = Amount entering - Amount leaving**
- The amount entering is simply the concentration of the pollutant, p, in the river times the flow rate of the river, f
- The amount leaving has the same flow rate, f
- Since the lake is assumed to be well-mixed, the concentration in the outflowing river will be equal to the concentration of the pollutant in the lake, $c(t)$
- The product $f\,c(t)$ gives the amount of pollutant leaving the lake per unit time

Pollution in a Lake

Differential Equations for Amount and Concentration of Pollutant

- The change in **amount of pollutant** satisfies the model
$$\frac{da(t)}{dt} = f\,p - f\,c(t)$$
- Since the lake maintains a constant volume V, then $c(t) = a(t)/V$, which also implies that $c'(t) = a'(t)/V$
- Dividing the above differential equation by the volume V,
$$\frac{dc(t)}{dt} = \frac{f}{V}(p - c(t))$$
- If the lake is initially clean, then $c(0) = 0$

Pollution in a Lake

Soluton of the Differential Equation: Rewrite the differential equation for the concentration of pollutant as
$$\frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0$$

- This DE should remind you of Newton's Law of Cooling with f/V acting like k and p acting like T_e
- Make the substitution, $z(t) = c(t) - p$, so $z'(t) = c'(t)$
- The initial condition becomes $z(0) = c(0) - p = -p$
- The initial value problem in $z(t)$ becomes,
$$\frac{dz(t)}{dt} = -\frac{f}{V}z(t), \quad \text{with} \quad z(0) = -p$$

Pollution in a Lake

Soluton of the Differential Equation (cont): Since

$$\frac{dz(t)}{dt} = -\frac{f}{V}z(t), \quad \text{with} \quad z(0) = -p$$

- The solution to this problem is

$$z(t) = -p\,e^{-\frac{ft}{V}} = c(t) - p$$

- $$c(t) = p\left(1 - e^{-\frac{ft}{V}}\right)$$

- The exponential decay in this solution shows

$$\lim_{t\to\infty} c(t) = p$$

- This is exactly what you would expect, as the entering river has a concentration of p

Example: Pollution in a Lake

Example: Pollution in a Lake Part 1

- Suppose that you begin with a 10,000 m^3 clean lake
- Assume the river entering has a flow of 100 m^3/day and the concentration of some pesticide in the river is measured to have a concentration of 5 ppm (parts per million)
- Form the differential equation describing the concentration of pollutant in the lake at any time t and solve it
- Find out how long it takes for this lake to have a concentration of 2 ppm

Example: Pollution in a Lake

Solution: This example follows the model derived above, so the differential equation for the concentration of pollutant is

$$\frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0$$

- Since $V = 10,000$, $f = 100$, and $p = 5$,

$$\frac{dc(t)}{dt} = -\frac{100}{10000}(c(t) - 5) \quad \text{with} \quad c(0) = 0$$

- Let $z(t) = c(t) - 5$, then the differential equation becomes,

$$\frac{dz}{dt} = -0.01z(t), \quad \text{with} \quad z(0) = -5$$

- This has a solution

$$z(t) = -5e^{-0.01t} = c(t) - 5$$

Example: Pollution in a Lake

Solution (cont): The concentration of pollutant in the lake is

$$c(t) = 5\left(1 - e^{-0.01t}\right)$$

- To find how long it takes for the concentration to reach 2 ppm, solve the equation

$$2 = 5 - 5e^{-0.01t)}$$

- Thus,

$$e^{-0.01t)} = \tfrac{3}{5} \quad \text{or} \quad e^{0.01t)} = \tfrac{5}{3}$$

- Solving this for t, we obtain

$$t = 100\ln\left(\tfrac{5}{3}\right) = 51.1 \text{ days}$$

Example: Pollution in a Lake 4

Example: Pollution in a Lake Part 2

- Suppose that when the concentration reaches 4 ppm, the pesticide is banned
- For simplicity, assume that the concentration of pesticide drops immediately to zero in the river
- Assume that the pesticide is not degraded or lost by any means other than dilution
- Find how long until the concentration reaches 1 ppm

Example: Pollution in a Lake 5

Solution: The new initial value problem becomes

$$\frac{dc}{dt} = -0.01(c(t) - 0) = -0.01c(t) \quad \text{with} \quad c(0) = 4$$

- This problem is in the form of a radioactive decay problem
- This has the solution

$$c(t) = 4e^{-0.01t}$$

- To find how long it takes for the concentration to return to 1 ppm, solve the equation

$$1 = 4e^{-0.01t} \quad \text{or} \quad e^{0.01t} = 4$$

- Solving this for t

$$t = 100\ln(4) = 138.6 \text{ days}$$

Pollution in a Lake: Complications

Pollution in a Lake: Complications The above discussion for pollution in a lake fails to account for many significant complications

- There are considerations of degradation of the pesticide, stratefication in the lake, and uptake and reentering of the pesticide through interaction with the organisms living in the lake
- The river will vary in its flow rate, and the leeching of the pesticide into river is highly dependent on rainfall, ground water movement, and rate of pesticide application
- Obviously, there are many other complications that would increase the difficulty of analyzing this model
- The next section shows numerical methods to handle more complicated models

Example: Lake Pollution with Evaporation 1

Example: Lake Pollution with Evaporation

- Suppose that a new industry starts up river from a lake at $t = 0$ days, and this industry starts dumping a toxic pollutant, $P(t)$, into the river at a rate of 7 g/day, which flows directly into the lake
- The flow of the river is 1000 m^3/day, which goes into the lake that maintains a constant volume of 400,000 m^3
- The lake is situated in a hot area and loses 50 m^3/day of water to evaporation (pure water with no pollutant), while the remainder of the water exits at a rate of 950 m^3/day through a river
- Assume that all quantities are well-mixed and that there are no time delays for the pollutant reaching the lake from the river

Example: Lake Pollution with Evaporation 2

Example: Lake Pollution with Evaporation (cont) Part a

- Write a differential equation that describes the concentration, $c(t)$, of the pollutant in the lake, using units of mg/m^3
- Solve the differential equation
- If a concentration of only 2 mg/m^3 is toxic to the fish population, then find how long until this level is reached
- If unchecked by regulations, then find what the eventual concentration of the pollutant is in the lake, assuming constant output by the new industry

Example: Lake Pollution with Evaporation 3

Solution: Let $P(t)$ be the amount of pollutant

The change in amount of pollutant =
Amount entering - Amount leaving

- The **change in amount** is $\frac{dP}{dt}$
- The concentration is given by $c(t) = P(t)/V$ and $c'(t) = P'(t)/V$
- The **amount entering** is the constant rate of pollutant dumped into the river, which is given by $k = 7000$ mg/day
- The **amount leaving** is given by the concentration of the pollutant in the lake, $c(t)$ (in mg/m^3), times the flow of water out of the lake, $f = 950$ m^3/day

Example: Lake Pollution with Evaporation 4

Solution (cont): The conservation of amount of pollutant is given by the equation:

$$\frac{dP}{dt} = k - f\,c(t) = 7000 - 950\,c(t)$$

- Evaporation concentrates the pollutant by allowing water to leave without the pollutant
- Divide the equation above by the volume, $V = 400,000$ m^3

$$\left(\frac{1}{V}\right)\frac{dP(t)}{dt} = \frac{k}{V} - \frac{f}{V}c(t) = \frac{7}{400} - \frac{950}{400000}c(t)$$

- The concentration equation is

$$\frac{dc}{dt} = \frac{7}{400} - \frac{950}{400000}c(t) = -\frac{f}{V}\left(c(t) - \frac{k}{f}\right)$$

Example: Lake Pollution with Evaporation 5

Solution (cont): The concentration equation is

$$\frac{dc}{dt} = -\frac{95}{40000}\left(c(t) - \frac{700}{95}\right)$$

- Make the change of variables, $z(t) = c(t) - \frac{700}{95}$, with $z(0) = -\frac{700}{95}$
- The differential equation is

$$\frac{dz}{dt} = -\frac{95}{40000}z(t) \quad \text{with} \quad z(0) = -\frac{700}{95}$$

- The solution is

$$z(t) = -\frac{700}{95}e^{-95t/40000} = c(t) - \frac{700}{95}$$

Example: Lake Pollution with Evaporation 6

Solution (cont): The concentration equation is

$$c(t) = \frac{700}{95}\left(1 - e^{-95t/40000}\right) \approx 7.368\left(1 - e^{-0.002375t}\right)$$

- If a concentration of 2 mg/m^3 is toxic to the fish population, then find when $c(t) = 2$ mg/m^3
- Solve

$$2 = 7.368\left(1 - e^{-0.002375t}\right) \quad \text{or} \quad e^{0.002375t} \approx 1.3726$$

- Thus, $t = \frac{\ln(1.3726)}{0.002375} \approx 133.3$ days
- The limiting concentration is

$$\lim_{t\to\infty} c(t) = \frac{700}{95} \approx 7.368$$

Example: Lake Pollution with Evaporation 7

Example: Lake Pollution with Evaporation (cont) Part b

- Suppose that the lake is at the limiting level of pollutant and a new environmental law is passed that shuts down the industry at a new time $t = 0$ days
- Write a new differential equation describing the situation following the shutdown of the industry and solve this equation
- Calculate how long it takes for the lake to return to a level that allows fish to survive

Example: Lake Pollution with Evaporation 8

Solution: Now $k = 0$, so the initial value problem becomes

$$\frac{dc}{dt} = -\frac{95}{40000}c(t) = -0.002375\,c(t) \quad \text{with} \quad c(0) = \frac{700}{95}$$

- This has the solution

$$c(t) = \frac{700}{95}\,e^{-0.002375t} \approx 7.368\,e^{-0.002375t}$$

- The concentration is reduced to 2 mg/m^3 when

$$2 = 7.368\,e^{-0.002375t} \quad \text{or} \quad e^{0.002375t} = 3.684$$

- The lake is sufficiently clean for fish when

$$t = \frac{\ln(3.684)}{0.002375} \approx 549 \text{ days}$$