Calculus for the Life Sciences II

Lecture Notes – Introduction to Differential Equations

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Outline

Introduction

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- Differential equations frequently arise in modeling situations

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- They describe population growth, chemical reactions, heat exchange, motion, and many other applications
- Differential equations are continuous analogs of discrete dynamical systems

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- In biology, a differential equation describes a growth rate, a reaction rate, or the change in some physiological state

Malthusian Growth

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The change in population between $(n+1)^{st}$ time and the n^{th} time is proportional to the population at the n^{th} time

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$$P(t + \Delta t) - P(t) = \Delta t \cdot rP(t)$$

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- Biologically, this equation says that the change (difference) in the population over a small period of time is found by taking the rate of growth times the population times the interval of time Δt
- The equation above can be rearranged to give

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} = rP(t)$$

Continuous Malthusian Growth

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- Take the limit of $\Delta t \to 0$, so

$$\lim_{\Delta t \to 0} \frac{P(t+\Delta t)-P(t)}{\Delta t} = \frac{dP(t)}{dt} = rP(t)$$

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- This is the **continuous Malthusian growth model**

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- Differentiating

$$\frac{dP(t)}{dt} = cre^{rt},$$

which is $rP(t)$, so satisfies the differential equation

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- Malthusian growth is often called exponential growth

Example: Malthusian Growth 1

Example: Malthusian Growth Consider the Malthusian growth model

$$\frac{dP(t)}{dt} = 0.02\,P(t) \quad \text{with} \quad P(0) = 100$$

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- Find the solution
- Determine how long it takes for this population to double

Example: Malthusian Growth 2

Solution: The solution is given by

$$P(t) = 100\, e^{0.02t}$$

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The population doubles when

$$\begin{aligned} 200 &= 100\, e^{0.02t} \\ 0.02t = \ln(2) \quad &\text{or} \quad t = 50\ln(2) \approx 34.66 \end{aligned}$$

Example 2: Malthusian Growth

Example 2: Suppose that a culture of *Escherichia coli* is growing according to the Malthusian growth model

$$\frac{dP(t)}{dt} = rP(t) \quad \text{with} \quad P(0) = 100,000$$

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- Compute the population after one hour

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- The population after one hour is

$$P(60) = 100,000e^{0.0277(60)} = 527,803$$

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- Like the Malthusian growth model, this has an exponential solution

$$R(t) = R_0 e^{-kt}$$

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- The general solution is

$$y(t) = c_1 \cos(kt) + c_2 \sin(kt),$$

where c_1 and c_2 are arbitrary constants

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This problem does not have an easily expressible solution

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- We solve this problem later in the semester

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- v is the voltage of the system, and a and b are constants

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- No explicit solution, but will study its behavior

Applications of Differential Equations 7

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- There are techniques for solving this

Applications of Differential Equations 8

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  - Malthusian and logistic growth and radioactive decay are 1^{st} order differential equations
  - Lotka-Volterra model is a 1^{st} order system of differential equations

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  - The Malthusian growth, radioactive decay, harmonic oscillator, and forced spring mass problem are linear differential equations
  - The swinging pendulum, van der Pol oscillator, logistic growth, and Lotka-Volterra model are nonlinear differential equations

Spring-Mass Problem 1

Spring-Mass Problem: Assume a mass attached to a spring without resistance satisfies the second order linear differential equation

$$y''(t) + 5y(t) = 0$$

Skip Example

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Show that two of the solutions to this differential equation are given by

$$y_1(t) = 3\sin\left(\sqrt{5}t\right) \qquad \text{and} \qquad y_2(t) = 2\cos\left(\sqrt{5}t\right)$$

Spring-Mass Problem 2

Solution: Undamped spring-mass problem

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- Take two derivatives of $y_1(t) = 3\sin\left(\sqrt{5}t\right)$

$$y_1'(t) = 3\sqrt{5}\cos\left(\sqrt{5}t\right) \quad \text{and} \quad y_1''(t) = -15\sin\left(\sqrt{5}t\right)$$

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- Substituting into the differential equation

$$y_1'' + 5y_1 = -15\sin\left(\sqrt{5}t\right) + 5\left(3\sin\left(\sqrt{5}t\right)\right) = 0$$

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- Substituting into the differential equation

$$y_1'' + 5y_1 = -15\sin\left(\sqrt{5}t\right) + 5\left(3\sin\left(\sqrt{5}t\right)\right) = 0$$

- Take two derivatives of $y_2(t) = 2\cos\left(\sqrt{5}t\right)$

$$y_2'(t) = -2\sqrt{5}\sin\left(\sqrt{5}t\right) \quad \text{and} \quad y_2''(t) = -10\cos\left(\sqrt{5}t\right)$$

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Solution: Undamped spring-mass problem

- Take two derivatives of $y_1(t) = 3\sin\left(\sqrt{5}t\right)$

$$y_1'(t) = 3\sqrt{5}\cos\left(\sqrt{5}t\right) \quad \text{and} \quad y_1''(t) = -15\sin\left(\sqrt{5}t\right)$$

- Substituting into the differential equation

$$y_1'' + 5y_1 = -15\sin\left(\sqrt{5}t\right) + 5\left(3\sin\left(\sqrt{5}t\right)\right) = 0$$

- Take two derivatives of $y_2(t) = 2\cos\left(\sqrt{5}t\right)$

$$y_2'(t) = -2\sqrt{5}\sin\left(\sqrt{5}t\right) \quad \text{and} \quad y_2''(t) = -10\cos\left(\sqrt{5}t\right)$$

- Substituting into the differential equation

$$y_2'' + 5y_2 = -10\cos\left(\sqrt{5}t\right) + 5\left(2\cos\left(\sqrt{5}t\right)\right) = 0$$

Damped Spring-Mass Problem 1

Damped Spring-Mass Problem: Assume a mass attached to a spring with resistance satisfies the second order linear differential equation

$$y''(t) + 2y'(t) + 5y(t) = 0$$

Skip Example

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Show that one solution to this differential equation is

$$y_1(t) = 2\,e^{-t}\sin(2t)$$

Damped Spring-Mass Problem 2

Solution: Damped spring-mass problem

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Solution: Damped spring-mass problem

- The 1^{st} derivative of $y_1(t) = 2\,e^{-t}\sin(2t)$

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Damped Spring-Mass Problem 2

Solution: Damped spring-mass problem

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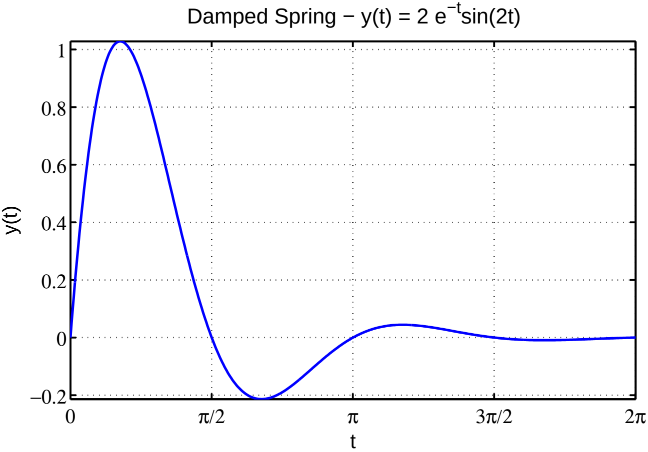
- Substitute into the spring-mass problem

$$\begin{aligned} y_1'' + 2y_1' + 5y &= -2e^{-t}(4\cos(2t) + 3\sin(2t)) \\ &\quad +2(2e^{-t}(2\cos(2t) - \sin(2t))) + 5(2e^{-t}\sin(2t)) \\ &= 0 \end{aligned}$$

Damped Spring-Mass Problem

Graph of Damped Oscillator



Evaporation Example 1

Evaporation Example: Animals lose moisture proportional to their surface area

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- If $V(t)$ is the volume of water in the animal, then the moisture loss satisfies the differential equation

$$\frac{dV}{dt} = -0.03\,V^{2/3}, \qquad V(0) = 8 \text{ cm}^3$$

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- Graph the solution

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Solution: Show $V(t) = (2 - 0.01t)^3$ satisfies

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- $V(0) = (2 - 0.01(0))^3 = 8$, so satisfies the initial condition
- Differentiate $V(t)$,

$$\frac{dV}{dt} = 3(2 - 0.01t)^2(-0.01) = -0.03(2 - 0.01t)^2$$

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- But $V^{2/3}(t) = (2 - 0.01t)^2$, so

$$\frac{dV}{dt} = -0.03\,V^{2/3}$$

Evaporation Example

Solution (cont): Find the time of total dessication

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- Thus,
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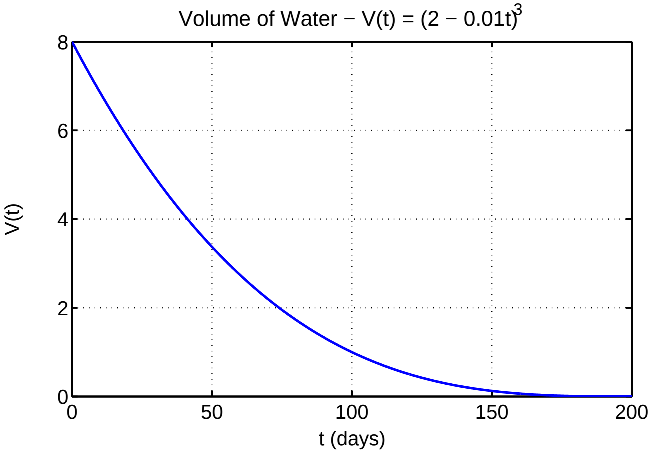
Evaporation Example

Solution (cont): Find the time of total dessication

- Must solve
$$V(t) = (2 - 0.01t)^3 = 0$$
- Thus,
$$2 - 0.01t = 0 \qquad \text{or} \qquad t = 200$$
- It takes 200 days for complete dessication

Evaporation Example

Graph of Dessication



Nonautonomous Example 1

Nonautonomous Example: Consider the nonautonomous differential equation with initial condition (**Initial Value Problem**):

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$$y(t) = \frac{2}{t^2 + 1}$$

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Solution: Consider the solution

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- Thus, the differential equation is satisfied

Nonautonomous Example

Solution of Nonautonomous Differentiation Equation