Calculus for the Life Sciences II

Lecture Notes – Integration by Substitution

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Outline

Introduction

Introduction: Managing More Integrals

- To date we have learned a collection of basic integrals
  - Polynomials
  - Power Law
  - Exponentials - e^{kt}
  - Trig Functions - $\sin(kt)$ and $\cos(kt)$

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- Apply to models using separable differential equations
  - The logistic growth model
  - Model for motion of an object subject to gravity

Logistic Growth Model for Yeast 1

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- After a lag period, the organisms begin growing according to **Malthusian growth**
- As the food source becomes limiting, the growth of the organism slows and the population levels off
- This behavior is modeled by adding a negative quadratic term to the Malthusian growth model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) \quad \text{with} \quad P(0) = P_0$$

Logistic Growth Model for Yeast

Experiment: G. F. Gause (*Struggle for Existence*) studied standard brewers yeast, *Saccharomyces cerevisiae*

Time (hr)	0	1.5	9	10	18	18	23
Volume	0.37	1.63	6.2	8.87	10.66	10.97	12.5

Time (hr)	25.5	27	34	38	42	45.5	47
Volume	12.6	12.9	13.27	12.77	12.87	12.9	12.7

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- *S. cerevisiae* placed in a closed vessel, where nutrient was changed regularly (every 3 hours)
- Simulates a constant source of nutrient

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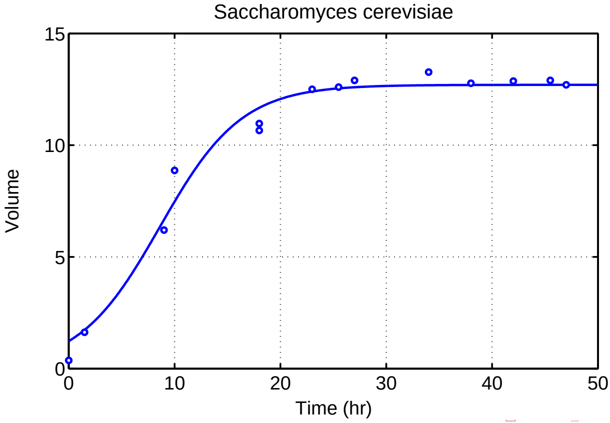
Logistic Growth Model for Yeast

Graph of data and best fitting model



Logistic Growth Model for Yeast 4

Model: The **Logistic Growth Model** that best fits the data is

$$\frac{dP}{dt} = 0.259\,P\left(1 - \frac{p}{12.7}\right), \quad \text{with} \quad P(0) = 1.23$$

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- The integral for P involves two integration techniques
- We'll concentrate on the the **integration by substitution**

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- Many functions that do not have an antiderivative
- **Integration by substitution** extends the number of integrable functions
- This technique is the inverse of the chain rule of differentiation
- The substitution technique is to find a function that reduces an integral to an easier form

Example 1

Example 1: Let a be a constant and consider the integral

$$\int (x+a)^n dx$$

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$$\int (x+a)^n dx = \int u^n du$$

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Make the substitution $u = x + a$, and the derivative gives the differentials $du = dx$, so

$$\begin{aligned} \int (x+a)^n dx &= \int u^n du \\ &= \frac{u^{n+1}}{n+1} + C \end{aligned}$$

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- There are a few choices that are very natural for a substitution
  - Let u be any expression of x in the exponent of the exponential function e or the argument of any trigonometric functions or the logarithm function
  - Let u be an expression of x inside parentheses raised to a power, where you should be able to see the derivative of that expression multiplying this expression to a power

Return to Logistic Growth 1

Return to Logistic Growth: The **Logistic Growth Model** is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) = -rP\left(\frac{P}{M} - 1\right)$$

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$$\int \frac{dP}{P\left(\frac{P}{M} - 1\right)} = -r\int dt$$

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- The integral on the left requires a technique from algebra
  - Fraction is split into two simple fractions (reverse of a common denominator)

$$\frac{1}{P\left(\frac{P}{M} - 1\right)} = \frac{\frac{1}{M}}{\left(\frac{P}{M} - 1\right)} - \frac{1}{P}$$

Return to Logistic Growth 2

Separated Differential Equation: From fractional form above, write the integral as

$$\int \frac{dP}{P\left(\frac{P}{M}-1\right)} = \frac{1}{M}\int \frac{dP}{\left(\frac{P}{M}-1\right)} - \int \frac{dP}{P}$$

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- For the other make the substitution $u = \frac{P}{M} - 1$, so $du = \frac{dP}{M}$

$$\frac{1}{M}\int \frac{dP}{\left(\frac{P}{M}-1\right)} = \int \frac{du}{u} = \ln|u| = \ln\left|\frac{P}{M}-1\right|$$

Return to Logistic Growth

Separated Differential Equation:

$$\int \frac{dP}{P\left(\frac{P}{M} - 1\right)} = -r \int dt = -rt + C$$

Return to Logistic Growth 3

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$$\ln\left|\frac{P}{M}-1\right| - \ln|P| = -rt + C$$

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- Exponentiating,

$$\left|\frac{P(t)-M}{MP(t)}\right| = e^{-rt+C}$$

Return to Logistic Growth

Solution: Removing the absolute value

$$\frac{P(t) - M}{MP(t)} = Ae^{-rt}$$

Return to Logistic Growth 4

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- Inserting this into the solution above gives

$$P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-rt}}$$

Return to Logistic Growth

Yeast Model: The best fitting yeast model

$$\frac{dP}{dt} = 0.259\,P\left(1 - \frac{p}{12.7}\right), \quad \text{with} \quad P(0) = 1.23$$

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$$P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-rt}}$$

- It follows that

$$P(t) = \frac{15.62}{1.23 + 11.47\,e^{-0.259\,t}}$$

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- It follows that

$$P(t) = \frac{15.62}{1.23 + 11.47\, e^{-0.259\, t}}$$

- This function creates the standard **S-shaped curve** of logistic growth and has the **carrying capacity** of **12.7**

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$$\int x^2 \cos(4 - x^3)dx$$

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$$u = 4 - x^3 \qquad \text{so} \qquad du = -3x^2 dx$$

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The solution of the integral is

$$\begin{aligned}\int x^2\cos(4-x^3)dx &= -\tfrac{1}{3}\int \cos(4-x^3)(-3x^2)dx \\ &= -\tfrac{1}{3}\int \cos(u)du \\ &= -\tfrac{1}{3}\sin(u)+C \\ &= -\tfrac{1}{3}\sin(4-x^3)+C\end{aligned}$$

Integration Example 2

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The solution of the integral is

$$\begin{aligned} \int \frac{(\ln(2x))^2}{x} dx &= \int u^2 du \\ &= \frac{u^3}{3} + C \\ &= \tfrac{1}{3}(\ln(2x))^3 + C \end{aligned}$$

Differential Equation Example 1

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$$\frac{dy}{dt} = \frac{2ty}{t^2+4}, \qquad y(0) = 8$$

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The right integral uses the substitution $u = t^2 + 4$, so $du = 2t\,dt$

$$\ln|y(t)| = \int \frac{du}{u} = \ln|u| + C = \ln(t^2+4) + C$$

Differential Equation Example 1

Solution (cont): The integrations give

$$\ln|y(t)| = \ln(t^2+4) + C$$

Differential Equation Example 1 2

Solution (cont): The integrations give

$$\ln|y(t)| = \ln(t^2 + 4) + C$$

- Exponentiating

$$y(t) = e^{\ln(t^2+4)+C} = e^C(t^2 + 4)$$

Differential Equation Example 1 2

Solution (cont): The integrations give

$$\ln|y(t)| = \ln(t^2+4) + C$$

- Exponentiating

$$y(t) = e^{\ln(t^2+4)+C} = e^C(t^2+4)$$

- Note that e^C could be positive or negative depending on the initial condition

Differential Equation Example 1 2

Solution (cont): The integrations give

$$\ln|y(t)| = \ln(t^2+4) + C$$

- Exponentiating

$$y(t) = e^{\ln(t^2+4)+C} = e^C(t^2+4)$$

- Note that e^C could be positive or negative depending on the initial condition
- From the initial condition, $y(0) = 8$, it follows that

$$y(t) = 2(t^2+4)$$

Differential Equation Example 2

Differential Equation Example 2: Consider

$$\frac{dy}{dt} = 2t\,e^{t^2-y}, \qquad y(0) = 2$$

Skip Example

Differential Equation Example 2

Differential Equation Example 2: Consider

$$\frac{dy}{dt} = 2t\, e^{t^2 - y}, \qquad y(0) = 2$$

Skip Example

Solution: Rewrite the differential equation

$$\frac{dy}{dt} = 2t\, e^{t^2} e^{-y}$$

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Solution: Rewrite the differential equation

$$\frac{dy}{dt} = 2t\, e^{t^2} e^{-y}$$

Separate the differential equation into the two integrals

$$\int e^y\, dy = \int 2t\, e^{t^2}\, dt$$

Differential Equation Example 2

Solution (cont): The right integral uses the substitution $u = t^2$, so $du = 2t\,dt$

$$\int e^y dy = e^y = \int 2t\, e^{t^2} dt = \int e^u du = e^u + C$$

Differential Equation Example 2 (2)

Solution (cont): The right integral uses the substitution $u = t^2$, so $du = 2t\,dt$

$$\int e^y dy = e^y = \int 2t\, e^{t^2} dt = \int e^u du = e^u + C$$

- By substitution the implicit solution is

$$e^y = e^{t^2} + C$$

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Solution (cont): The right integral uses the substitution $u = t^2$, so $du = 2t\,dt$

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- Taking logarithms

$$y(t) = \ln\left(e^{t^2} + C\right)$$

Differential Equation Example 2

2

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- By substitution the implicit solution is

$$e^y = e^{t^2} + C$$

- Taking logarithms

$$y(t) = \ln\left(e^{t^2} + C\right)$$

- From the initial condition, $y(0) = 2 = \ln(1 + C)$, it follows that

$$y(t) = \ln\left(e^{t^2} + e^2 - 1\right)$$

Logistic Growth

Logistic Growth: Suppose that a population of animals satisfies the logistic growth equation

$$\frac{dP}{dt} = 0.01\,P\left(1 - \frac{P}{2000}\right), \qquad P(0) = 50$$

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- Determine how long it takes for this population to double
- Find how long it takes to reach half of the carrying capacity

Logistic Growth 2

Solution: We separate this logistic growth model

$$\int \frac{dP}{P\left(\frac{P}{2000} - 1\right)} = -0.01 \int dt = -0.01\, t + C$$

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$$\int \frac{dP}{P\left(\frac{P}{2000}-1\right)} = -0.01\int dt = -0.01\,t + C$$

- The **Fundamental Theorem Algebra** gives

$$\frac{1}{P\left(\frac{P}{2000}-1\right)} = \frac{\frac{1}{2000}}{\left(\frac{P}{2000}-1\right)} - \frac{1}{P}$$

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$$\frac{1}{P\left(\frac{P}{2000}-1\right)} = \frac{\frac{1}{2000}}{\left(\frac{P}{2000}-1\right)} - \frac{1}{P}$$

- We use the substitution $u = \frac{P}{2000} - 1$, so $du = \frac{du}{2000}$

$$\frac{1}{2000}\int \frac{dP}{\left(\frac{P}{2000}-1\right)} - \int \frac{dP}{P} = \int \frac{du}{u} - \int \frac{dP}{P} = -0.01\,t + C$$

Logistic Growth

Solution (cont): From the substitution $u = \frac{P}{2000}$

$$\int \frac{du}{u} - \int \frac{dP}{P} = -0.01\,t + C$$

Logistic Growth

Solution (cont): From the substitution $u = \frac{P}{2000}$

$$\int \frac{du}{u} - \int \frac{dP}{P} = -0.01\,t + C$$

- Thus,

$$\ln|u| - \ln|P| = \ln\left|\frac{P - 2000}{2000}\right| - \ln|P| = -0.01\,t + C$$

Logistic Growth 3

Solution (cont): From the substitution $u = \frac{P}{2000}$

$$\int \frac{du}{u} - \int \frac{dP}{P} = -0.01\,t + C$$

- Thus,

$$\ln|u| - \ln|P| = \ln\left|\frac{P-2000}{2000}\right| - \ln|P| = -0.01\,t + C$$

- So,

$$\ln\left|\frac{P-2000}{2000\,P}\right| = -0.01\,t + C$$

Logistic Growth

Solution (cont): Exponentiating the previous expression

$$\frac{P(t) - 2000}{2000\,P(t)} = e^{-0.01\,t+C} = Ae^{-0.01\,t}$$

Logistic Growth

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- Solving for $P(t)$,

$$P(t) = \frac{2000}{1 - 2000A\,e^{-0.01\,t}}$$

Logistic Growth 4

Solution (cont): Exponentiating the previous expression

$$\frac{P(t) - 2000}{2000\,P(t)} = e^{-0.01\,t + C} = Ae^{-0.01\,t}$$

- Solving for $P(t)$,

$$P(t) = \frac{2000}{1 - 2000A\,e^{-0.01\,t}}$$

- With the initial condition, $P(0) = 50$,

$$P(t) = \frac{2000}{1 + 39\,e^{-0.01\,t}}$$

Logistic Growth

Solution (cont): The logistic growth model is

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- The population doubles when

$$P(t_d) = \frac{2000}{1 + 39\, e^{-0.01\, t_d}} = 100$$

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- Thus,

$$1 + 39\,e^{-0.01\,t_d} = 20 \qquad \text{or} \qquad e^{0.01\,t_d} = \frac{39}{19}$$

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- Solving for doubling time

$$t_d = 100\,\ln\left(\frac{39}{19}\right) = 71.9$$

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$$P(t) = \frac{2000}{1 + 39\, e^{-0.01\, t}}$$

Logistic Growth

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$$P(t_h) = \frac{2000}{1 + 39\, e^{-0.01\, t_h}} = 1000$$

- Thus,

$$1 + 39\, e^{-0.01\, t_h} = 2 \qquad \text{or} \qquad e^{0.01\, t_h} = 39$$

Logistic Growth

Solution (cont): The logistic growth model is

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- Thus,

$$1 + 39\,e^{-0.01\,t_h} = 2 \qquad \text{or} \qquad e^{0.01\,t_h} = 39$$

- Solving for doubling time

$$t_h = 100\,\ln(39) = 366.4$$

Escape Velocity 1

Escape Velocity: Find the velocity required to escape Earth's gravitation

Escape Velocity

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- Study the velocity of this object as it moves away from the surface of a planet using Newton's law of gravitational attraction, but ignoring any air resistence

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- Newton's law of gravitational attraction states that the force of attraction is inversely proportional to the square of the distance between the masses
- Newton's law of motion states that the mass of the object times the acceleration is equal to the sum of all the forces acting on the object

Escape Velocity 2

Gravitational Forces: An object of mass m is projected upward from Earth's surface with an initial velocity V_0

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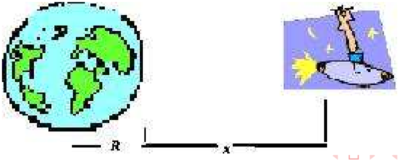
- Let x be the distance from the surface of the Earth, radius R
- Ignore air resistence
- Account for the variation of the Earth's gravitional field
- g is the acceleration of gravity at the surface of the Earth



Escape Velocity

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$$ma = -\frac{mgR^2}{(x+R)^2}$$

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- Need the velocity as a function of the distance rather than time
- Velocity $v = \frac{dx}{dt}$, where x is the distance from the Earth
- From the chain rule of differentiation

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Escape Velocity 4

Newton's Law of Motion: Differential equation for velocity

$$mv\frac{dv}{dx} = -\frac{mgR^2}{(x+R)^2}$$

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- Separation of variables gives

$$\int v\,dv = -\int \frac{gR^2}{(x+R)^2}dx$$

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- For the escape velocity, we must determine the smallest velocity required so that an object does not return to Earth
- Separation of variables gives

$$\int v\,dv = -\int \frac{gR^2}{(x+R)^2}dx$$

- Left hand side easily integrated, while right hand side requires a substitution $u = x + R$, so $du = dx$

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Separated Equation with $u = x + R$

$$\int v\,dv = -\int \frac{gR^2}{(x+R)^2}dx = -gR^2 \int u^{-2}du$$

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- Integrating

$$\frac{v^2}{2} = -gR^2\frac{u^{-1}}{-1} + C = \frac{gR^2}{x+R} + C$$

- Thus,

$$v^2 = \frac{2gR^2}{x+R} + 2C \qquad \text{or} \qquad v = \sqrt{\frac{2gR^2}{x+R} + 2C}$$

Escape Velocity 6

Solution: The initial condition gives $v(0) = V_0$, so

$$V_0^2 = \frac{2gR^2}{R} + 2C \qquad \text{or} \qquad 2C = V_0^2 - 2gR$$

Escape Velocity

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$$v^2(x) = \frac{2gR^2}{x+R} + V_0^2 - 2gR$$

- or

$$v(x) = \pm\sqrt{\frac{2gR^2}{x+R} + V_0^2 - 2gR}$$

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- The **escape velocity** is

$$V_0=\sqrt{2gR}$$

- For Earth, $g = 9.8$ m/sec^2 and $R = 6,378$ km, so the necessary V_0 for escape velocity is

$$V_0=\sqrt{2(0.0098)(6378)}=11.2 \text{ km/sec}$$

Lake Pollution with Seasonal Flow 1

Lake Pollution with Seasonal Flow Often the flow rate into a lake varies with the season

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- Suppose that a 200,000 m^3 lake maintains a constant volume and is initially clean
- A river flowing into the lake has 6 $\mu g/m^3$ of a pesticide
- Assume that the flow of the river has the sinusoidal form

$$f(t) = 100(2 - \cos(0.0172\, t)),$$

where t is in days

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- Graph the solution for 2 years

Lake Pollution with Seasonal Flow 2

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- The change in the amount of pesticide, $A(t)$, equals the amount entering - the amount leaving

$$\frac{dA(t)}{dt} = 600(2 - \cos(0.0172\,t)) - 100(2 - \cos(0.0172\,t))c(t)$$

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- Concentration satisfies $c(t) = \frac{A(t)}{200{,}000}$, so

$$\frac{dc}{dt} = -\frac{(2 - \cos(0.0172\,t))}{2000}(c - 6)$$

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- Separating variables

$$\int \frac{dc}{c-6} = -\frac{1}{2000}\int (2 - \cos(0.0172\,t))dt$$

Lake Pollution with Seasonal Flow 3

Solution: By letting $u = c - 6$ with $du = dc$, the integrals are

$$\int \frac{du}{u} = -0.0005 \int (2 - \cos(0.0172\,t))dt$$

Lake Pollution with Seasonal Flow 3

Solution: By letting $u = c - 6$ with $du = dc$, the integrals are

$$\int \frac{du}{u} = -0.0005 \int (2 - \cos(0.0172\,t))dt$$

- Integrating

$$\ln(u) = \ln(c(t) - 6) = -0.0005 \left(2t - \frac{\sin(0.0172\,t)}{0.0172} \right) + C$$

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- Integrating

$$\ln(u) = \ln(c(t) - 6) = -0.0005 \left(2t - \frac{\sin(0.0172\, t)}{0.0172} \right) + C$$

- By exponentiating this implicit solution, using the initial condition ($c(0) = 0$), and letting $\frac{1}{0.0172} = 58.14$, the solution becomes

$$c(t) = 6 \left(1 - e^{-0.0005(2t - 58.14\, \sin(0.0172\, t))} \right)$$

Lake Pollution with Seasonal Flow 4

Graph: Consider solution for 2 yr or 730 days

$$c(t) = 6\left(1 - e^{-0.0005(2t - 58.14\,\sin(0.0172\,t))}\right)$$