Calculus for the Life Sciences II

Lecture Notes – Integration by Substitution

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Outline

Introduction

Introduction: Managing More Integrals

- To date we have learned a collection of basic integrals
  - Polynomials
  - Power Law
  - Exponentials - e^{kt}
  - Trig Functions - $\sin(kt)$ and $\cos(kt)$
- **Integration by substitution** allows a substitution that reduces the integral to a simpler form
- This is basically this inverse of the **Chain Rule of differentiation**
- Apply to models using separable differential equations
  - The logistic growth model
  - Model for motion of an object subject to gravity

Logistic Growth Model for Yeast 1

Logistic Growth Model for Yeast: Model considers a limited food source

- After a lag period, the organisms begin growing according to **Malthusian growth**
- As the food source becomes limiting, the growth of the organism slows and the population levels off
- This behavior is modeled by adding a negative quadratic term to the Malthusian growth model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) \quad \text{with} \quad P(0) = P_0$$

Logistic Growth Model for Yeast

Experiment: G. F. Gause (*Struggle for Existence*) studied standard brewers yeast, *Saccharomyces cerevisiae*

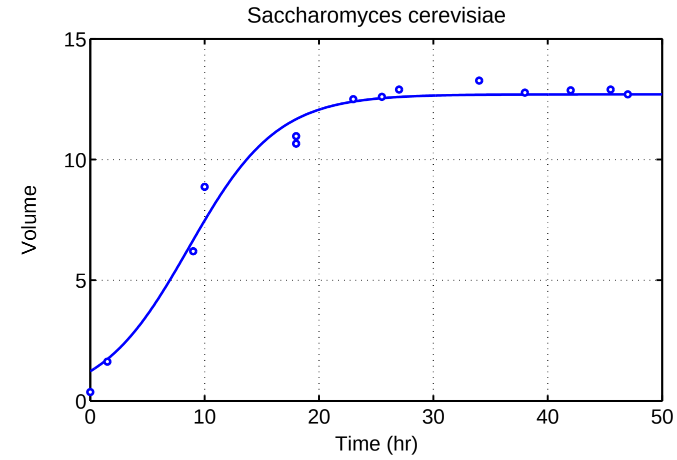
- *S. cerevisiae* placed in a closed vessel, where nutrient was changed regularly (every 3 hours)
- Simulates a constant source of nutrient

Time (hr)	0	1.5	9	10	18	18	23
Volume	0.37	1.63	6.2	8.87	10.66	10.97	12.5

Time (hr)	25.5	27	34	38	42	45.5	47
Volume	12.6	12.9	13.27	12.77	12.87	12.9	12.7

Logistic Growth Model for Yeast

Graph of data and best fitting model



Logistic Growth Model for Yeast

Model: The **Logistic Growth Model** that best fits the data is

$$\frac{dP}{dt} = 0.259\,P\left(1 - \frac{p}{12.7}\right), \quad \text{with} \quad P(0) = 1.23$$

- How do we find the solution to this nonlinear differential equation?
- This is a separable equation
- The integral for P involves two integration techniques
- We'll concentrate on the the **integration by substitution**

Integration by Substitution

Integration by Substitution

- Integration is the inverse of differentiation
- Many functions that do not have an antiderivative
- **Integration by substitution** extends the number of integrable functions
- This technique is the inverse of the chain rule of differentiation
- The substitution technique is to find a function that reduces an integral to an easier form

Example 1

Example 1: Let a be a constant and consider the integral

$$\int (x+a)^n dx$$

Make the substitution $u = x + a$, and the derivative gives the differentials $du = dx$, so

$$\begin{aligned}\int (x+a)^n dx &= \int u^n du \\ &= \frac{u^{n+1}}{n+1} + C \\ &= \frac{(x+a)^{n+1}}{n+1} + C\end{aligned}$$

Example 2

Example 2: Consider the integral

$$\int x\, e^{-x^2} dx$$

Make the substitution $u = -x^2$, and the derivative gives the differentials $du = -2x\, dx$, so

$$\begin{aligned}\int x\, e^{-x^2} dx &= \int e^{-x^2} \left(-\tfrac{1}{2}\right)(-2x) dx \\ &= -\tfrac{1}{2} \int e^u du \\ &= -\tfrac{1}{2} e^u + C \\ &= -\tfrac{1}{2} e^{-x^2} + C\end{aligned}$$

Example 3

Example 3: Consider the integral

$$\int \left(x^2 + 2x + 4\right)^3 (x+1) dx$$

Make the substitution $u = x^2 + 2x + 4$, and the derivative gives the differentials $du = (2x+2)dx$, so

$$\begin{aligned}\int \left(x^2+2x+4\right)^3 (x+1)dx &= \tfrac{1}{2} \int \left(x^2+2x+4\right)^3 (2x+2)dx \\ &= \tfrac{1}{2} \int u^3 du \\ &= \tfrac{u^4}{8} + C \\ &= \frac{\left(x^2+2x+4\right)^4}{8} + C\end{aligned}$$

Integration by Substitution

Integration by Substitution: What makes a good substitution?

- Choose u such that when u and du are substituted for the expression of x under the integrand, the remaining integral became of one of the basic integrals solved earlier
- There are a few choices that are very natural for a substitution
  - Let u be any expression of x in the exponent of the exponential function e or the argument of any trigonometric functions or the logarithm function
  - Let u be an expression of x inside parentheses raised to a power, where you should be able to see the derivative of that expression multiplying this expression to a power

Return to Logistic Growth 1

Return to Logistic Growth: The **Logistic Growth Model** is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) = -rP\left(\frac{P}{M} - 1\right)$$

- Separate Variables to give

$$\int \frac{dP}{P\left(\frac{P}{M} - 1\right)} = -r\int dt$$

- The integral on the right is very easy to solve
- The integral on the left requires a technique from algebra
  - Fraction is split into two simple fractions (reverse of a common denominator)

$$\frac{1}{P\left(\frac{P}{M} - 1\right)} = \frac{\frac{1}{M}}{\left(\frac{P}{M} - 1\right)} - \frac{1}{P}$$

Return to Logistic Growth 2

Separated Differential Equation: From fractional form above, write the integral as

$$\int \frac{dP}{P\left(\frac{P}{M} - 1\right)} = \frac{1}{M}\int \frac{dP}{\left(\frac{P}{M} - 1\right)} - \int \frac{dP}{P}$$

- One integral is easy

$$\int \frac{dP}{P} = \ln|P| + C$$

- For the other make the substitution $u = \frac{P}{M} - 1$, so $du = \frac{dP}{M}$

$$\frac{1}{M}\int \frac{dP}{\left(\frac{P}{M} - 1\right)} = \int \frac{du}{u} = \ln|u| = \ln\left|\frac{P}{M} - 1\right|$$

Return to Logistic Growth 3

Separated Differential Equation:

$$\int \frac{dP}{P\left(\frac{P}{M} - 1\right)} = -r\int dt = -rt + C$$

- From results above

$$\ln\left|\frac{P}{M} - 1\right| - \ln|P| = -rt + C$$

- Thus,

$$\ln\left|\frac{\frac{P}{M} - 1}{P}\right| = \ln\left|\frac{P - M}{MP}\right| = -rt + C$$

- Exponentiating,

$$\left|\frac{P(t) - M}{MP(t)}\right| = e^{-rt+C}$$

Return to Logistic Growth 4

Solution: Removing the absolute value

$$\frac{P(t) - M}{MP(t)} = Ae^{-rt}$$

- Solving for $P(t)$ gives

$$P(t) = \frac{M}{1 - MAe^{-rt}}$$

- With the initial condition, $P(0) = P_0$

$$P_0 = \frac{M}{1 - MA} \qquad \text{or} \qquad A = \frac{P_0 - M}{MP_0}$$

- Inserting this into the solution above gives

$$P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-rt}}$$

Return to Logistic Growth 5

Yeast Model: The best fitting yeast model

$$\frac{dP}{dt} = 0.259\,P\left(1 - \frac{p}{12.7}\right), \quad \text{with} \quad P(0) = 1.23$$

- The general logistic solution is

$$P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-rt}}$$

- It follows that

$$P(t) = \frac{15.62}{1.23 + 11.47\,e^{-0.259\,t}}$$

- This function creates the standard **S-shaped curve** of logistic growth and has the **carrying capacity** of **12.7**

Integration Example 1

Integration Example 1: Consider the integral

$$\int x^2\cos(4 - x^3)dx$$

Skip Example

Solution: A natural substitution is

$$u = 4 - x^3 \qquad \text{so} \qquad du = -3x^2 dx$$

The solution of the integral is

$$\begin{aligned}\int x^2\cos(4 - x^3)dx &= -\tfrac{1}{3}\int \cos(4 - x^3)(-3x^2)dx \\ &= -\tfrac{1}{3}\int \cos(u)du \\ &= -\tfrac{1}{3}\sin(u) + C \\ &= -\tfrac{1}{3}\sin(4 - x^3) + C\end{aligned}$$

Integration Example 2

Integration Example 2: Consider the integral

$$\int \frac{(\ln(2x))^2}{x}dx$$

Skip Example

Solution: A natural substitution is

$$u = \ln(2x) \qquad \text{so} \qquad du = \frac{dx}{x}$$

The solution of the integral is

$$\begin{aligned}\int \frac{(\ln(2x))^2}{x}dx &= \int u^2 du \\ &= \frac{u^3}{3} + C \\ &= \tfrac{1}{3}(\ln(2x))^3 + C\end{aligned}$$

Differential Equation Example 1 1

Differential Equation Example 1: Consider

$$\frac{dy}{dt} = \frac{2ty}{t^2 + 4}, \qquad y(0) = 8$$

Skip Example

Solution: Separate the differential equation into the two integrals

$$\int \frac{dy}{y} = \int \frac{2t}{t^2 + 4}dt$$

The right integral uses the substitution $u = t^2 + 4$, so $du = 2t\,dt$

$$\ln|y(t)| = \int \frac{du}{u} = \ln|u| + C = \ln(t^2 + 4) + C$$

Differential Equation Example 1 (2)

Solution (cont): The integrations give

$$\ln|y(t)| = \ln(t^2+4) + C$$

- Exponentiating

$$y(t) = e^{\ln(t^2+4)+C} = e^C(t^2+4)$$

- Note that e^C could be positive or negative depending on the initial condition
- From the initial condition, $y(0) = 8$, it follows that

$$y(t) = 2(t^2+4)$$

Differential Equation Example 2 (1)

Differential Equation Example 2: Consider

$$\frac{dy}{dt} = 2t\,e^{t^2-y}, \qquad y(0) = 2$$

Skip Example

Solution: Rewrite the differential equation

$$\frac{dy}{dt} = 2t\,e^{t^2}e^{-y}$$

Separate the differential equation into the two integrals

$$\int e^y\,dy = \int 2t\,e^{t^2}dt$$

Differential Equation Example 2 (2)

Solution (cont): The right integral uses the substitution $u = t^2$, so $du = 2t\,dt$

$$\int e^y\,dy = e^y = \int 2t\,e^{t^2}dt = \int e^u du = e^u + C$$

- By substitution the implicit solution is

$$e^y = e^{t^2} + C$$

- Taking logarithms

$$y(t) = \ln\left(e^{t^2} + C\right)$$

- From the initial condition, $y(0) = 2 = \ln(1+C)$, it follows that

$$y(t) = \ln\left(e^{t^2} + e^2 - 1\right)$$

Logistic Growth (1)

Logistic Growth: Suppose that a population of animals satisfies the logistic growth equation

$$\frac{dP}{dt} = 0.01\,P\left(1 - \frac{P}{2000}\right), \qquad P(0) = 50$$

- Find the general solution of this equation
- Determine how long it takes for this population to double
- Find how long it takes to reach half of the carrying capacity

Logistic Growth 2

Solution: We separate this logistic growth model

$$\int \frac{dP}{P\left(\frac{P}{2000}-1\right)} = -0.01\int dt = -0.01\,t + C$$

- The **Fundamental Theorem Algebra** gives

$$\frac{1}{P\left(\frac{P}{2000}-1\right)} = \frac{\frac{1}{2000}}{\left(\frac{P}{2000}-1\right)} - \frac{1}{P}$$

- We use the substitution $u = \frac{P}{2000} - 1$, so $du = \frac{du}{2000}$

$$\frac{1}{2000}\int \frac{dP}{\left(\frac{P}{2000}-1\right)} - \int \frac{dP}{P} = \int \frac{du}{u} - \int \frac{dP}{P} = -0.01\,t + C$$

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Lake Pollution with Seasonal Flow

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Differential Equations
Logistic Growth

Logistic Growth 3

Solution (cont): From the substitution $u = \frac{P}{2000}$

$$\int \frac{du}{u} - \int \frac{dP}{P} = -0.01\,t + C$$

- Thus,

$$\ln|u| - \ln|P| = \ln\left|\frac{P-2000}{2000}\right| - \ln|P| = -0.01\,t + C$$

- So,

$$\ln\left|\frac{P-2000}{2000\,P}\right| = -0.01\,t + C$$

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Logistic Growth 4

Solution (cont): Exponentiating the previous expression

$$\frac{P(t)-2000}{2000\,P(t)} = e^{-0.01\,t+C} = Ae^{-0.01\,t}$$

- Solving for $P(t)$,

$$P(t) = \frac{2000}{1 - 2000A\,e^{-0.01\,t}}$$

- With the initial condition, $P(0) = 50$,

$$P(t) = \frac{2000}{1 + 39\,e^{-0.01\,t}}$$

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Logistic Growth 5

Solution (cont): The logistic growth model is

$$P(t) = \frac{2000}{1 + 39\,e^{-0.01\,t}}$$

- The population doubles when

$$P(t_d) = \frac{2000}{1 + 39\,e^{-0.01\,t_d}} = 100$$

- Thus,

$$1 + 39\,e^{-0.01\,t_d} = 20 \qquad \text{or} \qquad e^{0.01\,t_d} = \frac{39}{19}$$

- Solving for doubling time

$$t_d = 100\,\ln\left(\frac{39}{19}\right) = 71.9$$

Logistic Growth

Solution (cont): The logistic growth model is

$$P(t) = \frac{2000}{1 + 39\,e^{-0.01\,t}}$$

- The population reaches half the carrying capacity when

$$P(t_h) = \frac{2000}{1 + 39\,e^{-0.01\,t_h}} = 1000$$

- Thus,

$$1 + 39\,e^{-0.01\,t_h} = 2 \qquad \text{or} \qquad e^{0.01\,t_h} = 39$$

- Solving for doubling time

$$t_h = 100\,\ln(39) = 366.4$$

Escape Velocity

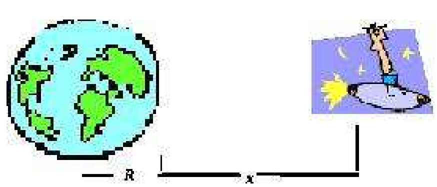
Escape Velocity: Find the velocity required to escape Earth's gravitation

- Consider an object shot away from a planet that is acted upon by only gravitational forces
- Study the velocity of this object as it moves away from the surface of a planet using Newton's law of gravitational attraction, but ignoring any air resistence
- Newton's law of gravitational attraction states that the force of attraction is inversely proportional to the square of the distance between the masses
- Newton's law of motion states that the mass of the object times the acceleration is equal to the sum of all the forces acting on the object

Escape Velocity

Gravitational Forces: An object of mass m is projected upward from Earth's surface with an initial velocity V_0

- Let x be the distance from the surface of the Earth, radius R
- Ignore air resistence
- Account for the variation of the Earth's gravitional field
- g is the acceleration of gravity at the surface of the Earth



Escape Velocity

Newton's Law of Motion:

$$ma = -\frac{mgR^2}{(x+R)^2}$$

- Acceleration is the time derivative of the velocity or $a = \frac{dv}{dt}$
- Need the velocity as a function of the distance rather than time
- Velocity $v = \frac{dx}{dt}$, where x is the distance from the Earth
- From the chain rule of differentiation

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Escape Velocity

Newton's Law of Motion: Differential equation for velocity

$$mv\frac{dv}{dx} = -\frac{mgR^2}{(x+R)^2}$$

- For the escape velocity, we must determine the smallest velocity required so that an object does not return to Earth
- Separation of variables gives

$$\int v\,dv = -\int \frac{gR^2}{(x+R)^2}dx$$

- Left hand side easily integrated, while right hand side requires a substitution $u = x + R$, so $du = dx$

Escape Velocity

Separated Equation with $u = x + R$

$$\int v\,dv = -\int \frac{gR^2}{(x+R)^2}dx = -gR^2\int u^{-2}du$$

- Integrating

$$\frac{v^2}{2} = -gR^2\frac{u^{-1}}{-1} + C = \frac{gR^2}{x+R} + C$$

- Thus,

$$v^2 = \frac{2gR^2}{x+R} + 2C \qquad \text{or} \qquad v = \sqrt{\frac{2gR^2}{x+R} + 2C}$$

Escape Velocity

Solution: The initial condition gives $v(0) = V_0$, so

$$V_0^2 = \frac{2gR^2}{R} + 2C \qquad \text{or} \qquad 2C = V_0^2 - 2gR$$

- Solution is

$$v^2(x) = \frac{2gR^2}{x+R} + V_0^2 - 2gR$$

- or

$$v(x) = \pm\sqrt{\frac{2gR^2}{x+R} + V_0^2 - 2gR}$$

Escape Velocity

Escape Velocity is the velocity at the surface of the planet, V_0, required for an object to escape the gravitational pull of a planet and not return

- The smallest velocity is $\lim_{x\to\infty} v(x) = 0$
- Thus,

$$\lim_{x\to\infty}\sqrt{\frac{2gR^2}{x+R} + V_0^2 - 2gR} = \sqrt{V_0^2 - 2gR} = 0$$

- The **escape velocity** is

$$V_0 = \sqrt{2gR}$$

- For Earth, $g = 9.8$ m/sec^2 and $R = 6,378$ km, so the necessary V_0 for escape velocity is

$$V_0 = \sqrt{2(0.0098)(6378)} = 11.2 \text{ km/sec}$$

Lake Pollution with Seasonal Flow 1

Lake Pollution with Seasonal Flow Often the flow rate into a lake varies with the season

- Suppose that a 200,000 m^3 lake maintains a constant volume and is initially clean
- A river flowing into the lake has 6 μg/m^3 of a pesticide
- Assume that the flow of the river has the sinusoidal form

$$f(t) = 100(2 - \cos(0.0172\,t)),$$

  where t is in days
- Find and solve the differential equation describing the concentration of the pesticide in the lake
- Graph the solution for 2 years

Lake Pollution with Seasonal Flow 2

Solution: Begin by creating the differential equation

- The change in the amount of pesticide, $A(t)$, equals the amount entering - the amount leaving

$$\frac{dA(t)}{dt} = 600(2 - \cos(0.0172\,t)) - 100(2 - \cos(0.0172\,t))c(t)$$

- Concentration satisfies $c(t) = \frac{A(t)}{200{,}000}$, so

$$\frac{dc}{dt} = -\frac{(2 - \cos(0.0172\,t))}{2000}(c - 6)$$

- Separating variables

$$\int \frac{dc}{c-6} = -\frac{1}{2000}\int (2 - \cos(0.0172\,t))dt$$

Lake Pollution with Seasonal Flow 3

Solution: By letting $u = c - 6$ with $du = dc$, the integrals are

$$\int \frac{du}{u} = -0.0005\int (2 - \cos(0.0172\,t))dt$$

- Integrating

$$\ln(u) = \ln(c(t) - 6) = -0.0005\left(2t - \frac{\sin(0.0172\,t)}{0.0172}\right) + C$$

- By exponentiating this implicit solution, using the initial condition $(c(0) = 0)$, and letting $\frac{1}{0.0172} = 58.14$, the solution becomes

$$c(t) = 6\left(1 - e^{-0.0005(2t - 58.14\,\sin(0.0172\,t))}\right)$$

Lake Pollution with Seasonal Flow 4

Graph: Consider solution for 2 yr or 730 days

$$c(t) = 6\left(1 - e^{-0.0005(2t - 58.14\,\sin(0.0172\,t))}\right)$$