Calculus for the Life Sciences II

Lecture Notes – Differentiation of Trigonometric Functions

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Outline

Introduction

Differentiation of Trigonometric Functions

- Showed **Sine** and **Cosine** Models were good for periodic phenomenon

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- Joseph Fourier (1768-1830) used series of trigonometric functions to approximate other phenomena, such as harmonic motion of vibrating strings
- Tidal flow results from the interaction of differing gravitational fields
  - The complex dynamics are approximated by a short series of trigonometric functions with periods related to the astronomical bodies causing the tidal flow

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- Most days there are two high tides (high-high and low-high) and two low tides (low-low and high-low)

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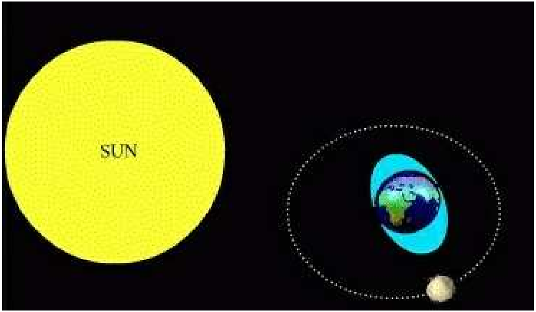
The primary forces generating the tides are from the gravity of the sun and the moon

Tides – Introduction

Gravitational forces of the Sun and Moon are the primary causes for generation of tides on Earth



Tidal Forces 1

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- Periodic motion of the moon about the Earth (about 25 hours) cause variations

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  - The influences of other planets

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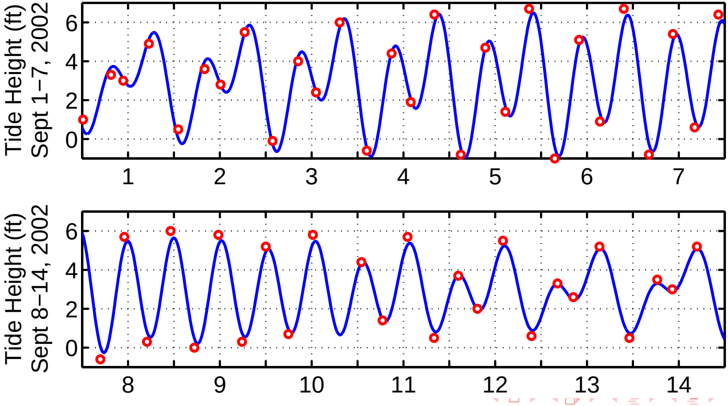
  - The data points indicate the actual values of the high and low tides from standard tide tables

Mathematical Model for Tides 2

First 2 weeks of Tides for San Diego in September 2002

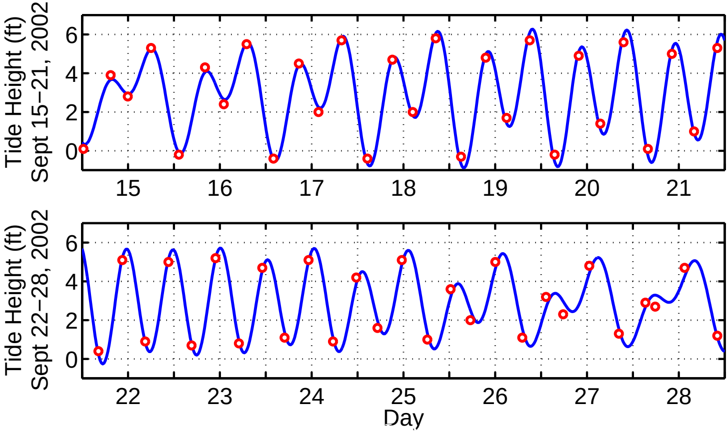
Model and Data



Mathematical Model for Tides 3

Last 2 weeks of Tides for San Diego in September 2002

Model and Data



Mathematical Model for Tides 4

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- Periods are fixed based on the rotations of the moon and Earth

Mathematical Model for Tides

Model for Height of Tides

$$h(t) = a_0 + \sum_{i=1}^{4} a_i \cos\left(\frac{2\pi}{p_i}(t - \phi_i)\right)$$

Mathematical Model for Tides

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- The parameters, a_i and ϕ_i, are fit using a least squares best fit to the high and low tides for the month of September 2002

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Force	Amplitude	Phase Shift
K_1	$a_1 = 0.878$	$\phi_1 = 16.246$
O_1	$a_2 = 0.762$	$\phi_2 = 14.311$
M_2	$a_3 = 1.993$	$\phi_3 = 6.164$
S_2	$a_4 = 0.899$	$\phi_4 = 10.857$

Mathematical Model for Tides

Model and Forces

New Moon	First Quarter	Full Moon	Last Quarter
			
September 6	September 13	September 21	September 29

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- The strongest force affecting the tides is the semidiurnal main lunar force
- The highest and lowest tides of the month coincide with the new moon and full moon

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Modelling Low and High Tides

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- The high and low tides should occur when $h'(t) = 0$

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Differentiation of Sine and Cosine

Differentiation of Sine Below is the graph of sine and its derivative

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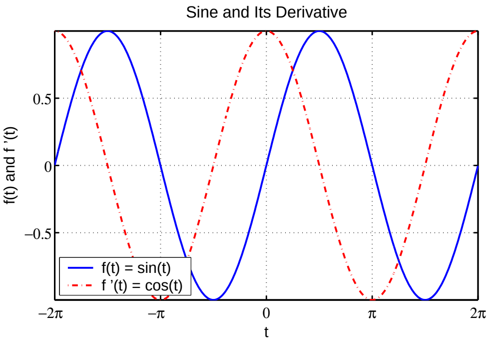
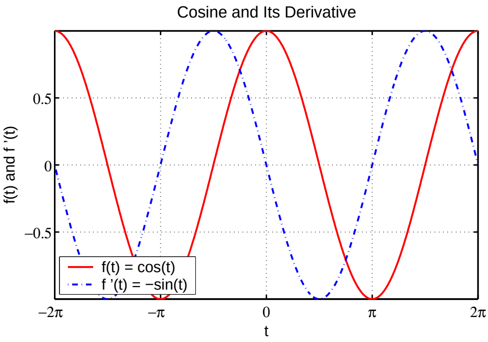


Differentiation of Sine and Cosine 3

Differentiation of Cosine Below is the graph of cosine and its derivative

$$\frac{d}{dx}\cos(x) = -\sin(x)$$



Differentiation of Sine and Cosine

General Rule of Differentiation of Sine and Cosine

Differentiation of Sine and Cosine 4

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Example 3: More Examples of Differentiation

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- Sine and cosine are equal when

$$t = \frac{\pi}{4} + n\pi$$

Example: Damped Oscillator 3

Solution (cont): Damped Oscillator satisfies

$$\begin{aligned} y(t) &= 2\,e^{-t}\sin(t) \\ y'(t) &= 2e^{-t}(\cos(t)-\sin(t)) \end{aligned}$$

Example: Damped Oscillator 3

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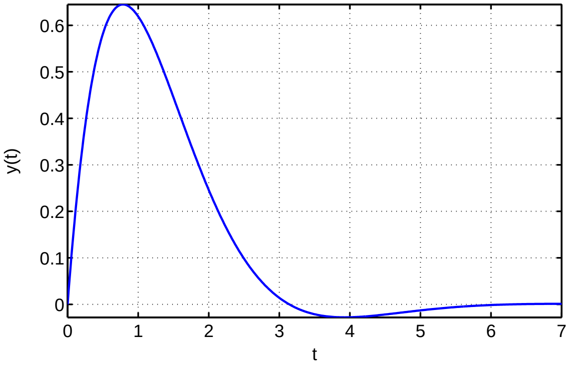
  - The minimum happens when $t=\frac{5\pi}{4}$ with

$$y\left(\frac{5\pi}{4}\right) = 2e^{-\frac{5\pi}{4}}\sin\left(\frac{5\pi}{4}\right) \approx -0.02786$$

Example: Damped Oscillator 4

Solution (cont): Damped Oscillator

$$y(t) = 2\,e^{-t}\sin(t)$$



High and Low Tides 1

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- The times and heights of the high and low tides use the local extrema

High and Low Tides

Model for Tides

$$h(t) = a_0 + \sum_{i=1}^{4} a_i \cos\left(\frac{2\pi}{p_i}(t - \phi_i)\right)$$

High and Low Tides 2

Model for Tides

$$h(t) = a_0 + \sum_{i=1}^{4} a_i \cos\left(\frac{2\pi}{p_i}(t - \phi_i)\right)$$

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- Clearly, this equation is too complicated to find the extrema by hand
- The Computer labs have shown that finding zeroes of this function are readily done using either Excel's Solver or Maple's fsolve command

High and Low Tides 3

New Moon There was a New moon on September 6, 2002

High and Low Tides

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- In the first week, the data show that the highest tide is 6.7 ft on Sept. 6, while the lowest tide is −1.0 ft on the same day
- So what does our model using four cosine functions predict to be the highest and lowest tides of this week?

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Low Tide Prediction for September 6, 2002

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- Parameters were fit for complete month of September and given earlier

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  - The lowest tide of the first week from the model occurs when $t_{min} = 124.58$ hrs with a $h(t_{min}) = -0.86$ ft

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High and Low Tides

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  - The model overshoots the tide height by about 0.14 feet and misses the time by 60 minutes

High and Low Tides

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- Repeat process for High Tide

High and Low Tides

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  - The highest tide of the first week from the model occurs when $t_{max} = 142.56$ hrs with a $h(t_{max}) = 6.40$ ft
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High and Low Tides

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High and Low Tides

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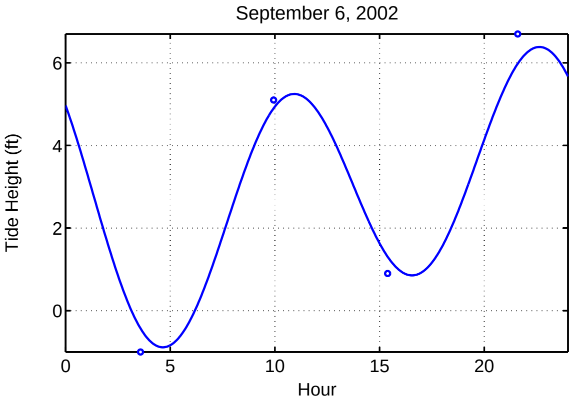
  - This corresponds to Sept. 6 at 10:33 pm
  - The actual high-high tide on Sept. 6 is 6.7 ft occurring at 9:36 pm
  - The model undershoots the tide height by about 0.3 feet and misses the time by 57 minutes

High and Low Tides

Graph of Tides: Model and Data



High and Low Tides

Summary of Tide Model for September 2002

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High and Low Tides 7

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- Model only uses four cosine functions to try to predict an entire month of high and low tides
- This is a reasonable approach to the problem
- Obviously, the addition of more trigonometric functions and more parameters can produce a much more accurate model
- Actual models use 12-14 trigonometric functions to model tides
- The information line (619-221-8824) for the San Diego Beach report gives tide information

Change in Temperature 1

Maximum Change in Temperature The sine function can be used to approximate the temperature during a day

$$T(t) = A + B\sin(\omega(t-\phi)),$$

with constants A, $B \geq 0$, $\omega > 0$, and $\phi \in [0, 24)$ are determined from the data

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- Suppose that the coolest temperature for a day occurs at **3 am** and is **56°F**
- Assume at **3 pm**, the hottest temperature of **82°F occurs**

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Maximum Change in Temperature The Temperature is modeled by

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Maximum Change in Temperature The Temperature is modeled by

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- Find the constants that best fit the data for the temperature during the day assuming that the temperature has a 24 hour period
- Determine the times during the day that the temperature is rising most rapidly and falling most rapidly
- Give the rate of change of temperature at those times

Change in Temperature 3

Solution: The temperature during a day

$$T(t) = A + B\sin(\omega(t-\phi))$$

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$$T(t) = A + B\sin(\omega(t - \phi))$$

- The average temperature is

$$A = (56 + 82)/2 = 69^\circ\mathrm{F}$$

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$$B = 82 - 69 = 13^\circ\text{F}$$

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$$B = 82 - 69 = 13^\circ\text{F}$$

- The 24 hour periodicity gives

$$24\omega = 2\pi \qquad \text{or} \qquad \omega = \frac{\pi}{12}$$

Change in Temperature

Solution (cont): The temperature during a day

$$T(t) = 69 + 13\sin\left(\frac{\pi}{12}(t-\phi)\right)$$

Change in Temperature

Solution (cont): The temperature during a day

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- The maximum occurs at 3 pm or $t = 15$

Change in Temperature 4

Solution (cont): The temperature during a day

$$T(t) = 69 + 13\sin\left(\frac{\pi}{12}(t-\phi)\right)$$

- The maximum occurs at 3 pm or $t = 15$
- The maximum of the sine function occurs at $\frac{\pi}{2}$

Change in Temperature

Solution (cont): The temperature during a day

$$T(t) = 69 + 13\sin\left(\frac{\pi}{12}(t-\phi)\right)$$

- The maximum occurs at 3 pm or $t = 15$
- The maximum of the sine function occurs at $\frac{\pi}{2}$
- The phase shift, ϕ, solves

$$\frac{\pi}{12}(15-\phi) = \frac{\pi}{2}$$

Change in Temperature

Solution (cont): The temperature during a day

$$T(t) = 69 + 13\sin\left(\tfrac{\pi}{12}(t-\phi)\right)$$

- The maximum occurs at 3 pm or $t = 15$
- The maximum of the sine function occurs at $\frac{\pi}{2}$
- The phase shift, ϕ, solves

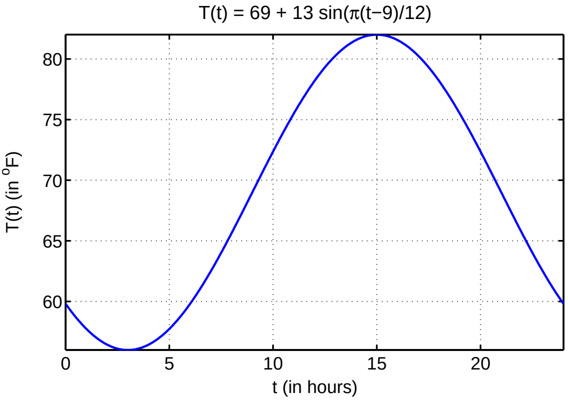
$$\frac{\pi}{12}(15-\phi) = \frac{\pi}{2}$$

- It follows that

$$15 - \phi = 6 \qquad \text{or} \qquad \phi = 9 \text{ hr}$$

Change in Temperature

Graph of Temperature Model



Change in Temperature 6

Solution (cont): Graph shows temperature is rising most rapidly in the morning and falling most rapidly in the evening

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Model is

$$T(t) = 69 + 13\sin\left(\frac{\pi}{12}(t-9)\right)$$

Change in Temperature 6

Solution (cont): Graph shows temperature is rising most rapidly in the morning and falling most rapidly in the evening

Model is

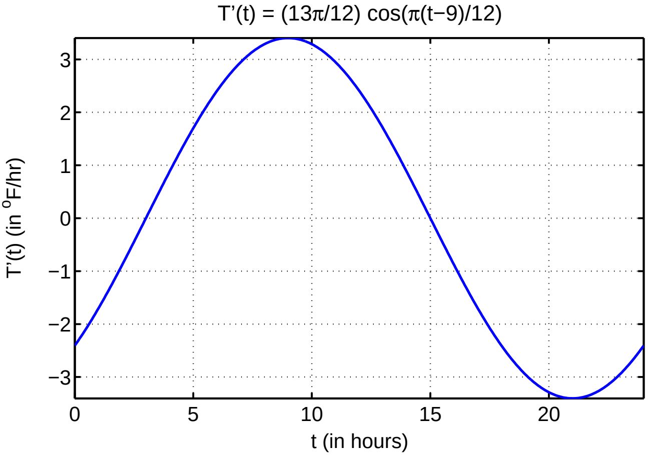
$$T(t) = 69 + 13\sin\left(\tfrac{\pi}{12}(t-9)\right)$$

The derivative satisfies

$$T'(t) = \frac{13\pi}{12}\cos\left(\tfrac{\pi}{12}(t-9)\right)$$

Change in Temperature

Graph of Derivative of Temperature Model



Change in Temperature

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Change in Temperature

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Change in Temperature

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- Find the maximum rate of change by properties of $T'(t)$
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- Maximum increase at 9 am with

$$T'(9) = \frac{13\pi}{12}\cos(0) = \frac{13\pi}{12} \approx 3.4^\circ\mathrm{F/hr}$$

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Solution (cont): The derivative is

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$$T'(9) = \frac{13\pi}{12}\cos(0) = \frac{13\pi}{12} \approx 3.4^\circ\text{F/hr}$$

- Minimum increase 12 hours later with $t = 21$, so

$$T'(21) = \frac{13\pi}{12}\cos(\pi) = -\frac{13\pi}{12} \approx -3.4^\circ\text{F/hr}$$

Change in Temperature

Alternate Solution: The maximum and minimum rate of change occurs when second derivative is zero

$$T''(t) = -\frac{13\pi^2}{144} \sin\left(\tfrac{\pi}{12}(t-9)\right)$$

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- Thus,
$$t = 9 + 12\,n \qquad n = 0, 1, ...$$

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- This gives same result as before