Calculus for the Life Sciences II

Lecture Notes – Differentiation of Trigonometric Functions

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Outline

Introduction

Differentiation of Trigonometric Functions

- Showed Sine and Cosine Models were good for periodic phenomenon
- Trigonometric functions are used to approximate more complicated behavior
- Joseph Fourier (1768-1830) used series of trigonometric functions to approximate other phenomena, such as harmonic motion of vibrating strings
- Tidal flow results from the interaction of differing gravitational fields
  - The complex dynamics are approximated by a short series of trigonometric functions with periods related to the astronomical bodies causing the tidal flow

Tides – Introduction

- Oceans are one of the great remaining frontiers
- Over half of the population on this planet lives within 100 miles of the oceans
- Tidal flow affects the daily dynamics of coastal zones
  - The Bay of Fundy in Newfoundland has tides rising over 16 meters in a 6.25 hour time period
  - In San Diego, the tidal flow is not so dramatic
- The tides do affect a variety of marine behaviors
- Most days there are two high tides (high-high and low-high) and two low tides (low-low and high-low)

Tides – Introduction

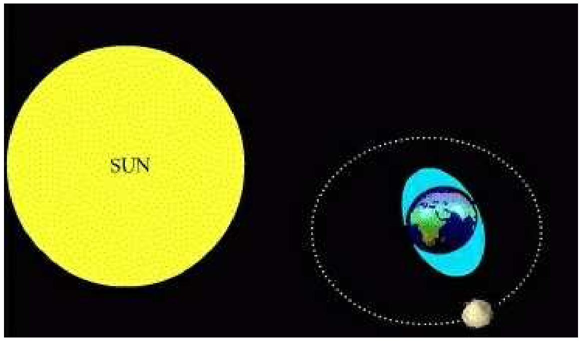
How are tides predicted and what is the basis for their variability?

What causes the changes in amplitude and period between the high and low tides?

The primary forces generating the tides are from the gravity of the sun and the moon

Tides – Introduction

Gravitational forces of the Sun and Moon are the primary causes for generation of tides on Earth



Tidal Forces 1

Four Dominant Tidal Forces

- Diurnal components about 24 hours (once per day)
  - K_1, the lunisolar force
  - O_1, the main lunar force
- Semidiurnal components about 12 hours (twice per day)
  - M_2, the main lunar force
  - S_2, the main solar force
- Periodic motion of the moon about the Earth (about 25 hours) cause variations

Tidal Forces 2

Tidal Forces

- When the moon, Earth, and sun align at either a full moon or a new moon, then the tides are at their highest and lowest as the forces of gravity enhance tidal flow
- The tides show the least variation when the moon is in its first or last quarter
- Other complications include
  - The elliptical orbit of the moon around the Earth
  - The elliptical orbit of the Earth around the sun
  - The influences of other planets

Mathematical Model for Tides 1

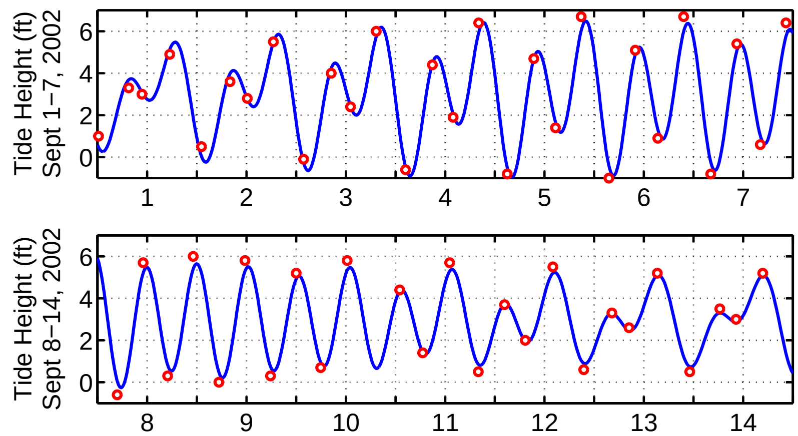
Mathematical Model for Tides

- **What mathematical tools can help predict the tides?**
- Use a series of trigonometric functions to approximate the behavior of the tides
- Standard programs use 12-14 trigonometric functions
- Next Slide are graphs of the high and low tides for San Diego for September 2002
  - The model is generated using only four trigonometric functions from the four forces described above
  - The data points indicate the actual values of the high and low tides from standard tide tables

Mathematical Model for Tides 2

First 2 weeks of Tides for San Diego in September 2002

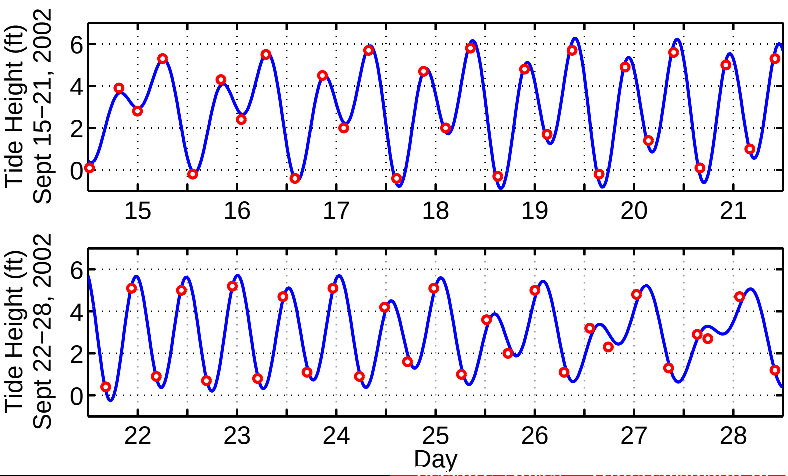
Model and Data



Mathematical Model for Tides 3

Last 2 weeks of Tides for San Diego in September 2002

Model and Data



Mathematical Model for Tides 4

Model for Height of Tides, $h(t)$ in feet with t hours from midnight 1^{st} day of the month

- The function $h(t)$ is formed by the sum of four cosine functions and a constant
- The periods of the cosine functions reflect the periodic nature of the forces
  - K_1, lunisolar diurnal force with period $p_1 = 23.934$
  - O_1, main lunar diurnal force with period $p_2 = 25.819$
  - M_2, main lunar semidiurnal force with period $p_3 = 12.421$
  - S_2, main solar semidiurnal force with period $p_4 = 12.00$
- Periods are fixed based on the rotations of the moon and Earth

Mathematical Model for Tides 5

Model for Height of Tides

- The amplitudes associated with each force are $a_i, i = 1, .., 4$
- The phase shifts associated with each force are $\phi_i, i = 1, .., 4$
- A vertical shift satisfies a_0
- The parameters, a_i and ϕ_i, are fit using a least squares best fit to the high and low tides for the month of September 2002

$$h(t) = a_0 + \sum_{i=1}^{4} a_i \cos\left(\frac{2\pi}{p_i}(t - \phi_i)\right)$$

Mathematical Model for Tides 6

$$h(t) = a_0 + \sum_{i=1}^{4} a_i \cos\left(\frac{2\pi}{p_i}(t - \phi_i)\right)$$

Best Fitting Parameters

Vertical Shift $a_0 = 2.937$ ft

Force	Amplitude	Phase Shift
K_1	$a_1 = 0.878$	$\phi_1 = 16.246$
O_1	$a_2 = 0.762$	$\phi_2 = 14.311$
M_2	$a_3 = 1.993$	$\phi_3 = 6.164$
S_2	$a_4 = 0.899$	$\phi_4 = 10.857$

Mathematical Model for Tides 7

Model and Forces

- The strongest force affecting the tides is the semidiurnal main lunar force
- The highest and lowest tides of the month coincide with the new moon and full moon

New Moon	First Quarter	Full Moon	Last Quarter
			
September 6	September 13	September 21	September 29

Mathematical Model for Tides 8

Modelling Low and High Tides

- **When do the highest and lowest tides occur based on the mathematical model?**
- The high and low points of a function are the maxima and minima
- This uses differentiation of our model, $h(t)$
- The high and low tides should occur when $h'(t) = 0$

Differentiation of Sine and Cosine 1

Differentiation of Sine and Cosine

- The derivative of these functions is found using the definition of the derivative and some trigonometric identities
- **Derivative of Sine**

$$\frac{d}{dx}\sin(x) = \cos(x)$$

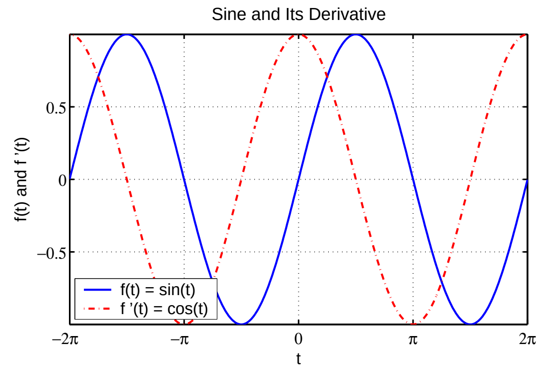
- **Derivative of Cosine**

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

Differentiation of Sine and Cosine 2

Differentiation of Sine Below is the graph of sine and its derivative

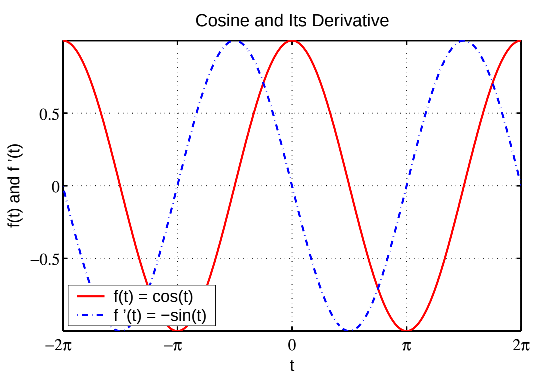
$$\frac{d}{dx}\sin(x) = \cos(x)$$



Differentiation of Sine and Cosine 3

Differentiation of Cosine Below is the graph of cosine and its derivative

$$\frac{d}{dx}\cos(x) = -\sin(x)$$



Differentiation of Sine and Cosine 4

General Rule of Differentiation of Sine and Cosine

- The **chain rule** can be applied to give a more general rule of differentiation
- **General Derivative of Sine**

$$\frac{d}{dx}\sin(f(x)) = f'(x)\cos(f(x))$$

- **General Derivative of Cosine**

$$\frac{d}{dx}\cos(f(x)) = -f'(x)\sin(f(x))$$

Example 1: Derivative of Sine Function

Example 1: Consider the function

$$f(x) = \sin(x^2 + 1)$$

Find the derivative of $f(x)$

Skip Example

Solution: Since the derivative of $x^2 + 1$ is $2x$, the derivative of $f(x)$ is

$$f'(x) = 2x\cos(x^2 + 1)$$

Example 2: Derivative of Cosine Function

Example 2: Consider the function

$$f(x) = e^{-3x}\cos(x^2 + 4)$$

Find the derivative of $f(x)$

Skip Example

Solution: This derivative uses the product and chain rule

$$\begin{aligned} f'(x) &= e^{-3x}(-2x\sin(x^2+4)) + \cos(x^2+4)(e^{-3x}(-3)) \\ f'(x) &= -e^{-3x}(2x\sin(x^2+4) + 3\cos(x^2+4)) \end{aligned}$$

Example 3: More Examples of Differentiation

Example 3: Consider the function

$$f(x) = 3x^2\sin(\ln(x + 2))$$

Find the derivative of $f(x)$

Skip Example

Solution: This derivative uses the product and chain rule

$$\begin{aligned} f'(x) &= (3x^2)\left(\frac{d}{dx}\sin(\ln(x+2))\right) + 6x\sin(\ln(x+2)) \\ f'(x) &= \frac{3x^2\cos(\ln(x+2))}{x+2} + 6x\sin(\ln(x+2)) \end{aligned}$$

Example 4: More Examples of Differentiation

Example 4: Consider the function

$$f(x) = 4e^{-\cos(2x+1)}$$

Find the derivative of $f(x)$

Skip Example

Solution: This derivative uses the chain rule

$$g'(x) = 4e^{-\cos(2x+1)}(2\sin(2x+1))$$

Example: Damped Oscillator 1

Example: Damped Oscillator

Consider the function

$$y(t) = 2\,e^{-t}\sin(t)$$

Skip Example

- Function describes the motion of a damped oscillator, like shock absorbers on a car
- Find the absolute maximum and minimum for this function for $t \geq 0$
- Graph of this function

Example: Damped Oscillator 2

Solution: Damped Oscillator is given by

$$y(t) = 2\,e^{-t}\sin(t)$$

- Derivative found with the **product rule**

$$\begin{aligned} y'(t) &= 2(e^{-t}\cos(t) + e^{-t}(-1)\sin(t)) \\ &= 2e^{-t}(\cos(t) - \sin(t)) \end{aligned}$$

- The extrema occur when $f'(t) = 0$, which happens whenever

$$\cos(t) = \sin(t)$$

- Sine and cosine are equal when

$$t = \frac{\pi}{4} + n\pi$$

Example: Damped Oscillator 3

Solution (cont): Damped Oscillator satisfies

$$\begin{aligned} y(t) &= 2\,e^{-t}\sin(t) \\ y'(t) &= 2e^{-t}(\cos(t) - \sin(t)) \end{aligned}$$

- The exponential function damps this solution to zero
  - Horizontal asymptote of $y = 0$
- The function is **zero** whenever $t = n\pi$ for n an integer
- The absolute maximum and minimum occur at the first relative maximum and minimum
  - The maximum occurs when $t = \frac{\pi}{4}$ with

$$y\left(\tfrac{\pi}{4}\right) = 2e^{-\frac{\pi}{4}}\sin\left(\tfrac{\pi}{4}\right) \approx 0.6448$$

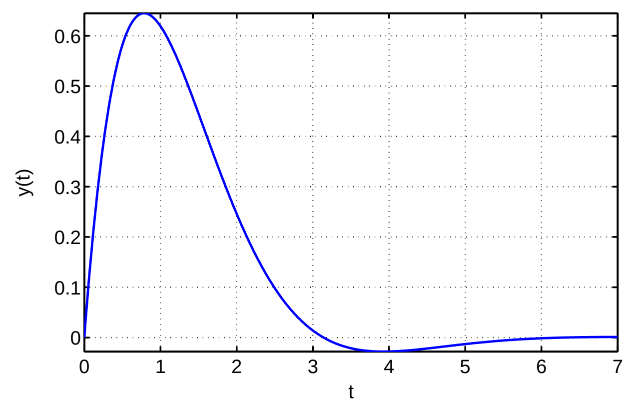
  - The minimum happens when $t = \frac{5\pi}{4}$ with

$$y\left(\tfrac{5\pi}{4}\right) = 2e^{-\frac{5\pi}{4}}\sin\left(\tfrac{5\pi}{4}\right) \approx -0.02786$$

Example: Damped Oscillator 4

Solution (cont): Damped Oscillator

$$y(t) = 2\,e^{-t}\sin(t)$$



High and Low Tides 1

High and Low Tides

- The highest and lowest tides of the month occur near the Full or New moon phases
- The gravity of the moon assists the gravity of the sun to enlarge the tides
- Use the model to predict the highest high-high tide and lowest low-low tide for the first week
- Determine the error between the model and the actual values for these tides
- The times and heights of the high and low tides use the local extrema

High and Low Tides 2

Model for Tides

$$h(t) = a_0 + \sum_{i=1}^{4} a_i \cos\left(\frac{2\pi}{p_i}(t - \phi_i)\right)$$

The derivative satisfies:

$$h'(t) = -\sum_{i=1}^{4} \left(\frac{2\pi a_i}{p_i}\right) \sin\left(\frac{2\pi}{p_i}(t - \phi_i)\right)$$

- Clearly, this equation is too complicated to find the extrema by hand
- The Computer labs have shown that finding zeroes of this function are readily done using either Excel's Solver or Maple's fsolve command

High and Low Tides 3

New Moon There was a New moon on September 6, 2002

- The graphs show many local extrema for the month of September
- Usually four of them each day
- Localize the search for the extrema using the visual information from the graph
- In the first week, the data show that the highest tide is 6.7 ft on Sept. 6, while the lowest tide is -1.0 ft on the same day
- So what does our model using four cosine functions predict to be the highest and lowest tides of this week?

High and Low Tides 4

Low Tide Prediction for September 6, 2002

- Parameters were fit for complete month of September and given earlier
- Set $h'(t) = 0$ and solved with a computer
- Low Tide Prediction
  - The lowest tide of the first week from the model occurs when $t_{min} = 124.58$ hrs with a $h(t_{min}) = -0.86$ ft
  - This corresponds to Sept. 6 at 4:35 am
  - The actual low-low tide on Sept. 6 is -1.0 ft occurring at 3:35 am
  - The model overshoots the tide height by about 0.14 feet and misses the time by 60 minutes

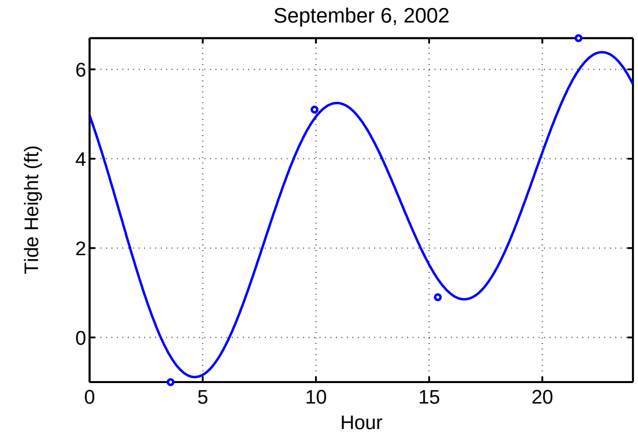
High and Low Tides 5

High Tide Prediction for September 6, 2002

- Repeat process for High Tide
- High Tide Prediction
  - The highest tide of the first week from the model occurs when $t_{max} = 142.56$ hrs with a $h(t_{max}) = 6.40$ ft
  - This corresponds to Sept. 6 at 10:33 pm
  - The actual high-high tide on Sept. 6 is 6.7 ft occurring at 9:36 pm
  - The model undershoots the tide height by about 0.3 feet and misses the time by 57 minutes

High and Low Tides 6

Graph of Tides: Model and Data



High and Low Tides 7

Summary of Tide Model for September 2002

- The calculations above show that our model introduces a moderate error
- Model only uses four cosine functions to try to predict an entire month of high and low tides
- This is a reasonable approach to the problem
- Obviously, the addition of more trigonometric functions and more parameters can produce a much more accurate model
- Actual models use 12-14 trigonometric functions to model tides
- The information line (619-221-8824) for the San Diego Beach report gives tide information

Change in Temperature 1

Maximum Change in Temperature The sine function can be used to approximate the temperature during a day

$$T(t) = A + B\sin(\omega(t-\phi)),$$

with constants A, $B \geq 0$, $\omega > 0$, and $\phi \in [0, 24)$ are determined from the data

- Suppose that the coolest temperature for a day occurs at **3 am** and is **56°F**
- Assume at **3 pm**, the hottest temperature of **82°F occurs**

Change in Temperature 2

Maximum Change in Temperature The Temperature is modeled by

$$T(t) = A + B\sin(\omega(t-\phi)),$$

- Find the constants that best fit the data for the temperature during the day assuming that the temperature has a 24 hour period
- Determine the times during the day that the temperature is rising most rapidly and falling most rapidly
- Give the rate of change of temperature at those times

Change in Temperature 3

Solution: The temperature during a day

$$T(t) = A + B\sin(\omega(t-\phi))$$

- The average temperature is

$$A = (56+82)/2 = 69^\circ\text{F}$$

- The amplitude of this function is found from the difference between the high temperature and the average temperature

$$B = 82 - 69 = 13^\circ\text{F}$$

- The 24 hour periodicity gives

$$24\omega = 2\pi \qquad \text{or} \qquad \omega = \frac{\pi}{12}$$

Change in Temperature 4

Solution (cont): The temperature during a day

$$T(t) = 69 + 13\sin\left(\tfrac{\pi}{12}(t-\phi)\right)$$

- The maximum occurs at 3 pm or $t = 15$
- The maximum of the sine function occurs at $\frac{\pi}{2}$
- The phase shift, ϕ, solves

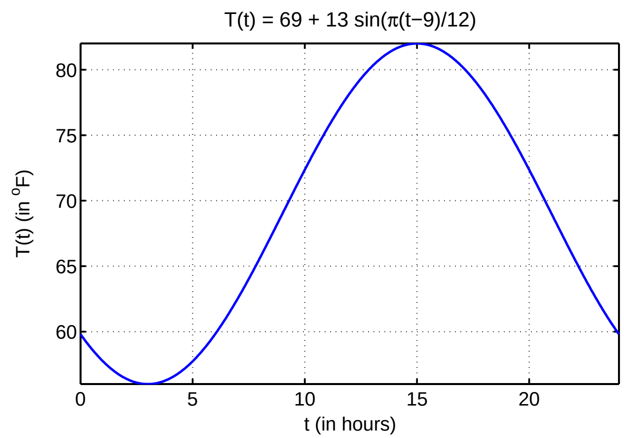
$$\frac{\pi}{12}(15-\phi) = \frac{\pi}{2}$$

- It follows that

$$15 - \phi = 6 \qquad \text{or} \qquad \phi = 9 \text{ hr}$$

Change in Temperature 5

Graph of Temperature Model



Change in Temperature

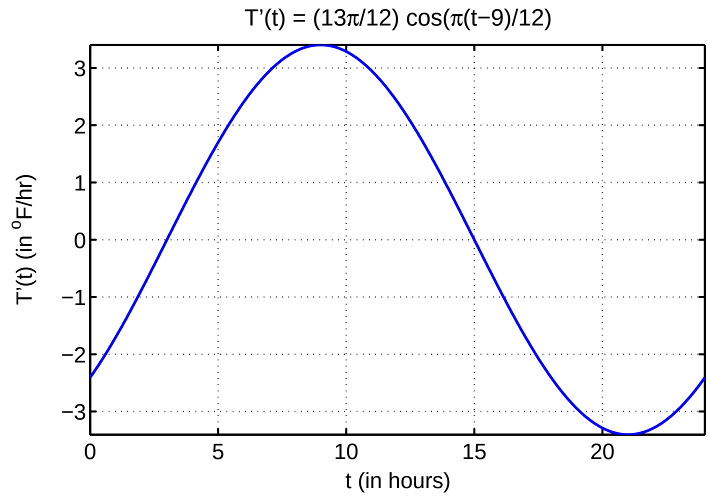
Solution (cont): Graph shows temperature is rising most rapidly in the morning and falling most rapidly in the evening

Model is
$$T(t) = 69 + 13\sin\left(\tfrac{\pi}{12}(t-9)\right)$$

The derivative satisfies
$$T'(t) = \frac{13\pi}{12}\cos\left(\tfrac{\pi}{12}(t-9)\right)$$

Change in Temperature

Graph of Derivative of Temperature Model



Change in Temperature

Solution (cont): The derivative is
$$T'(t) = \frac{13\pi}{12}\cos\left(\tfrac{\pi}{12}(t-9)\right)$$

- Find the maximum rate of change by properties of $T'(t)$
- Cosine has maximum when argument is zero, when $t = 9$
- Maximum increase at 9 am with
$$T'(9) = \frac{13\pi}{12}\cos(0) = \frac{13\pi}{12} \approx 3.4^\circ\text{F/hr}$$
- Minimum increase 12 hours later with $t = 21$, so
$$T'(21) = \frac{13\pi}{12}\cos(\pi) = -\frac{13\pi}{12} \approx -3.4^\circ\text{F/hr}$$

Change in Temperature

Alternate Solution: The maximum and minimum rate of change occurs when second derivative is zero
$$T''(t) = -\frac{13\pi^2}{144}\sin\left(\tfrac{\pi}{12}(t-9)\right)$$

- The sine function is zero when the argument is $n\pi$
- Solve
$$\frac{\pi}{12}(t-9) = n\pi, \qquad n = 0, 1, \dots$$
- Thus,
$$t = 9 + 12n \qquad n = 0, 1, \dots$$
- This gives same result as before