May 17, 2005 Math 122a Name Key

1. (25pts) Evaluate the following integrals:

a. $\int \left(9\sin(3x) + \frac{8x^2}{\sqrt{x^3+1}}\right) dx$, $= -3\cos(3x) + \frac{8}{3}\int 3x^2 (x^3+1)^{-1/2} dx$ $\quad u = x^3+1$, $du = 3x^2 dx$

$= -3\cos(3x) + \frac{8}{3}\int u^{-1/2} du$

11 Answer $-3\cos(3x) + \frac{16}{3}(x^3+1)^{1/2} + C$

b. $\int_0^3 \left(5e^{-x} + \frac{\ln(x+1)}{x+1}\right) dx$, $= -5e^{-x}\Big|_0^3 + \int_0^{\ln(4)} u\, du = -5(e^{-3}-1) + \frac{u^2}{2}\Big|_0^{\ln(4)}$

$u = \ln(x+1)$
$du = \frac{dx}{x+1}$

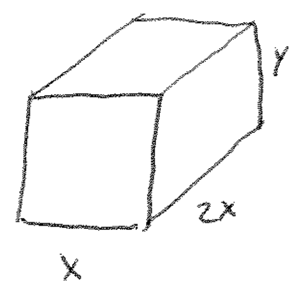
14 Answer $5(1-e^{-3}) + \frac{1}{2}(\ln(4))^2 = 4.75106 + 0.9609 = 5.71196$

2. (13pts) Your biological samples must be packed in closed box that uses expensive insulated material. The design requires that the length of the base is twice its width. You have 30,000 cm^2 of material. Draw a diagram for the problem, labeling clearly the unknown dimensions. Then find the dimensions of the box, which maximizes the volume. (Remember to include the top, bottom and four sides of the box.)

Constraint: $4x^2 + 6xy = 30000$

$3xy = 15000 - 2x^2$

$y = \frac{5000}{x} - \frac{2}{3}x$



$V(x) = 10000x - \frac{4}{3}x^3$

$V'(x) = 10000 - 4x^2 = 0 \Rightarrow x^2 = \frac{10000}{4} \Rightarrow x = 50$

$y = \frac{5000}{50} - \frac{2}{3}50$

Volume (from your diagram) = $2x^2y$

Surface Area (from your diagram) = $4x^2 + 6xy = 30000$

Length = 100 cm Width = 50 cm Height = $200/3$ cm

38

3. (25pts) Solve the following initial value problems:

a. $\dfrac{dy}{dt} = 5 + 0.2\,y, \quad y(0) = 7,$

$\frac{dy}{dt} = 0.2(y+25)$ $\qquad z(t) = y(t) + 25$

$z(0) = 32$

$\frac{dz}{dt} = 0.2z$

$z(t) = 32e^{0.2t}$

8 $\quad y(t) =$ $32e^{0.2t} - 25$

b. $\dfrac{dy}{dt} = 4\,t + 8, \quad y(0) = 10,$

$y(t) = \int (4t+8)\,dt = 2t^2 + 8t + C$

7 $\quad y(t) =$ $2t^2 + 8t + 10$

c. $\dfrac{dy}{dt} = \dfrac{1 - 2\,e^{-2t}}{y}, \quad y(0) = 2,$

$\int y\,dy = \int (1 - 2e^{-2t})\,dt$

$\frac{y^2}{2} = t + e^{-2t} + \frac{c}{2}$

$y^2(t) = 2t + 2e^{-2t} + c$

$y^2(0) = 4 = 2 + c \Rightarrow c = 2$

$y(t) =$ $(2t + 2e^{-2t} + 2)^{1/2}$

10

25

4. (18pts) Consider the curves $y = 9 - 3x$ and $y = x^2 - 3x$.

a. Find all the x and y-intercepts for both curves. Determine the slope of the line and the vertex of the parabola. Find the points of intersection, then sketch the graph of these curves. (Label the points clearly.)

Line: x-intercept $\underline{3}$ y-intercept $\underline{9}$ Slope $m = \underline{-3}$

Parabola: x-intercepts $\underline{0, \ 3}$ y-intercept $\underline{0}$ Vertex $\underline{(3/2, -9/4)}$

Points of Intersection: $\underline{(-3, 18)}$ and $\underline{(3, 0)}$

$9 - 3x = x^2 - 3x \Rightarrow$

$x^2 - 9 = 0$

$x = \pm 3$

GRAPH:

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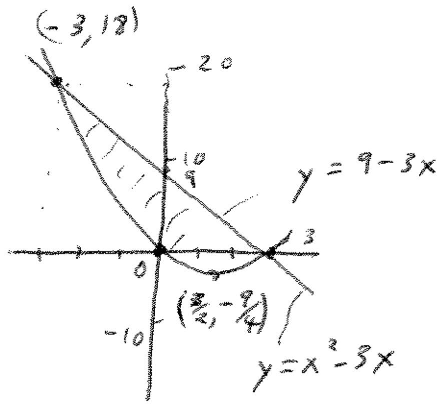


b. Set up and solve the integral that determines the closed area above the parabola and below the line.

$$\int_{-3}^{3} \left[(9-3x) - (x^2 - 3x) \right] dx = \int_{-3}^{3} (9 - x^2)\, dx = 9x - \frac{x^3}{3} \Big|_{-3}^{3}$$

$$= 27 - 9 - (-27 + 9)$$

8

Area = $\underline{36}$

18

5. (22pts) a. Testosterone is an important male hormone that regulates a number of male bodily functions, including the male sex drive. Levels of this hormone tend to peak in the early morning hours and reach low levels at night. However, there is a fair variation in this diurnal fluctuation. Suppose that testosterone levels are measured on a male subject, and it is found that this subject has a peak of 9.7 ng/ml at about 8 am ($t = 8$) and a low value of 7.9 ng/ml at about 8 pm ($t = 20$). Assume that the levels of testosterone can be modeled using the following function:

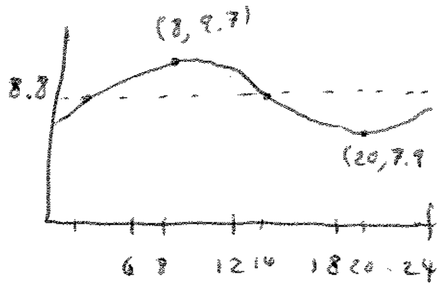
$$T(t) = A + B\cos(\omega(t - \phi)),$$

where A, B, ω, and ϕ are constants and t is in hours. Use the data above to find the four parameters, then sketch a graph for the level of testosterone in this male subject.

$A = \frac{9.7+7.9}{2}$

GRAPH:

$\omega \cdot 24 = 2\pi \Rightarrow \omega = \frac{\pi}{12}$



$A =$ 8.8

$B =$ 0.9

$\omega = \frac{\pi}{12} = 0.261799$

$\phi =$ 8

10 $T(t) = 8.8 + 0.9\cos\left(\frac{\pi}{12}(t-8)\right)$

b. Find the derivative of $T(t)$ and give its value at noon ($t = 12$).

$T'(t) = -\frac{3\pi}{40}\sin\left(\frac{\pi}{12}(t-8)\right) = -0.235624\sin(0.26180(t-8))$

4 $T'(12) = -0.20405$

c. Find the average level of testosterone in the male subject from midnight ($t = 0$) to noon ($t = 12$). The average level of testosterone is computed by the following integral:

8 $$\frac{1}{12}\int_0^{12} T(t)\,dt = \frac{1}{12}\int_0^{12}\left(8.8 + 0.9\cos\left(\frac{\pi}{12}(t-8)\right)\right)dt$$

$$= \frac{1}{12}\left(8.8t\Big|_0^{12} + \frac{10.8}{\pi}\sin\left(\frac{\pi}{12}(t-8)\right)\Big|_0^{12}\right) = 8.8 + 0.28648\left(\sin\left(\frac{\pi}{3}\right) - \sin\left(-\frac{2\pi}{3}\right)\right)$$

Average = 9.2962

22

6. (25pts) Ricker's model is often used to study the population of fish. Let P_n be the population of a species of fish in years n and suppose that Ricker's model is given by

$$P_{n+1} = R(P_n) = 7P_n e^{-0.002\,P_n}.$$

a. Assume that the initial population is $P_0 = 50$, then determine the population of fish for the next two years (P_1 and P_2).

3

$P_1 =$ 316.69 $\quad P_2 =$ 1176.69

b. Find $R'(P)$, then determine the maximum of this function (both P and $R(P)$ values). Sketch a graph of $R(P)$ with the identity function for $P \geq 0$, showing the intercepts and any horizontal asymptotes.

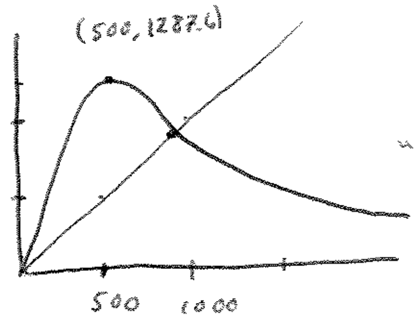
P-intercept 0 $\quad R$-intercept 0 $\quad$ Horizontal Asymptote $R =$ 0

$R'(P) = 7e^{-0.002P}(1 - 0.002P)$

$P_{max} =$ 500 $\quad R(P_{max}) =$ 1287.58

GRAPH:

12



c. Find all equilibria for Ricker's model and determine the stability of the equilibria. Justify your stability argument by evaluating the derivative of the updating function.

10

$P_{1e} =$ 0 $\quad R'(P_{1e}) =$ 7

Stable or **Unstable** $\quad$ **Monotonic** or Oscillatory

$P_{1e} =$ 972.96 $\quad R'(P_{1e}) =$ −0.9459

Stable or Unstable $\quad$ Monotonic or **Oscillatory**

$P_e = 7P_e e^{-0.002P_e}$

$P_e = 0$ or

$e^{0.002P_e} = 7$

$P_e = 500 \ln(7)$

25

7. (18pts) a. An initially clean lake ($c(0) = 0$) maintains a constant volume of $V = 10^5$ m^3 of water. There are two streams entering this lake with differing concentrations of agricultural pesticide. The first stream has a flow rate of $f_1 = 200$ m^3/day with a pesticide concentration of $Q_1 = 7\,\mu$g/m^3. A second stream has a flow rate of $f_2 = 300$ m^3/day with a pesticide concentration of $Q_2 = 2\,\mu$g/m^3. Assume that this is a well-mixed lake with a stream flowing out at a rate of $f_3 = 500$ m^3/day (with the pesticide in that stream equal to the concentration in the lake). Write a differential equation describing the concentration of pesticide in the lake ($c(t)$) and solve this differential equation.

$$\frac{dA}{dt} = f_1 Q_1 + f_2 Q_2 - (f_1 + f_2)c = 1400 + 600 - 500c$$

$$\frac{dc}{dt} = \frac{2000}{10^5} - \frac{500}{10^5}c$$

$z(t) = c(t) - 4$

$z(0) = -4$

$\frac{dz}{dt} = -0.005z$

$z(t) = -4e^{-0.005t}$

$\frac{dc}{dt} =$ $0.02 - 0.005c = -0.005(c-4)$

13

$c(t) =$ $4 - 4e^{-0.005t}$

b. Determine how long until the lake has a concentration of $2\,\mu$g/m^3 of pesticide. Also, find the limiting concentration of pesticide in this lake. Sketch a graph of the solution.

$4 - 4e^{-0.005t} = 2$

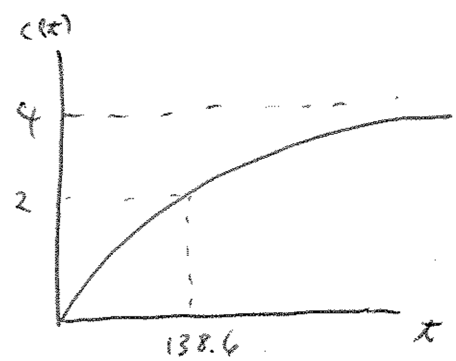
$4e^{-0.005t} = 2$

$e^{0.005t} = 2$

$t = 200\ln(2)$

$c(t_2) = 2$ when $t_2 =$ 138.63 days Limiting Concentration = $4\ \mu g/m^3$

5



18

8. (24pts) One of the hazards of modern medicine is the possible exposure to radioactive sources. Radioactive cobalt, ^{57}Co, is used for sterilization of medical equipment and treatment of some cancers.

a. Radioactive cobalt, ^{57}Co, has a half-life of 270 days. A differential equation describing the radioactive decay of ^{57}Co is given by

$$\frac{dR}{dt} = -kR, \qquad R(0) = 10,$$

where t is in days. Solve this differential equation and find the value of k. (Use at least **4 significant figures** for k.)

$5 = 10e^{-270k}$

$\Rightarrow k = \frac{\ln(2)}{270}$

$k =$ 0.0025672 $R(t) =$ $10\, e^{-0.0025672t}$

6

b. Suppose a technician is receiving an exposure from mislaid sample of radioactive cobalt, ^{57}Co

$$C(t) = 0.2e^{-kt},$$

in mCi/day. The total exposure over 200 days is given by the integral

$$\int_0^{200} C(t)\,dt. = \frac{0.2}{-0.0025672} e^{-0.0025672t}\Big|_0^{200}$$

Find this total exposure.

$= 77.906(1 - e^{-0.5134})$

6

$\int_0^{200} C(t)\,dt =$ 31.285 mCi

c. Use the Midpoint rule, dividing up the interval $t \in [0, 200]$ into 5 even intervals to approximate the integral in Part b. What is the percent error between this approximation and the actual value from Part b?

$\int_0^{200} C(t)dt \approx 40[C(20) + C(60) + C(100) + C(140) + C(180)]$

$= 40(0.94995 + 0.85724 + 0.77358 + 0.69809 + 0.62996)(0.2) = 31.2706$

6

Midpoint Aprox. $\int_0^{200} C(t)\,dt \simeq$ 31.2706 mCi Percent Error = -0.046%

d. How long can the technician stay near this source if the exposure is to be kept to less than 10 mCi?

$\int_0^t 0.2\, e^{-0.0025672t}dt = 77.906(1 - e^{-0.0025672t}) = 10$

$\frac{77.906}{67.906} = e^{0.0025672t} = 1.14726$

6

Exposure time is less than 53.513 days.

$t = \frac{\ln(1.14726)}{0.0025672} = 53.513$

24

9. (30pts) a. The U. S. population was 76.0 million in 1900 and 105.7 million in 1920. Use the Malthusian growth model

$$\frac{dP}{dt} = rP \qquad P(t) = 76e^{rt}$$

to represent the population of the U. S. Solve this differential equation, assuming that $t = 0$ is 1900 and with the data above find the value of r to 4 significant figures. How long does it take for the population to double with this model.

$105.7 = 76e^{20r}$

6 $\quad r =$ 0.01649 $\quad P(t) =$ $76.0e^{0.01649t}$ $\quad r = \frac{1}{20}\ln\left(\frac{105.7}{76}\right) =$

Doubling Time = 42.025 years $\quad 152 = 76e^{rt_d} \quad t_d = \frac{\ln(2)}{r}$

b. The population of U. S. was 151.3 million in 1950. Find the percent error between the Malthusian growth model prediction and the census data.

2 $\quad P(50) =$ 173.37M $\quad$ Percent Error = 14.59%

c. A logistic growth model for the U. S. population that reasonably fits the census data for the 20^{th} century is given by

$$\frac{dP}{dt} = 0.02P\left(1 - \frac{P}{420}\right), \qquad P(0) = 76.0.$$

where t is in years after 1900. Write Euler's formula for solving this differential equation, then use Euler's method with a stepsize of $h = 1$ on the interval $0 \le t \le 3$ to approximate the solution $P(t)$ at times $t = 1$, 2, and 3.

$t_0 = 0 \quad P_0 = 76$

$t_1 = 1 \quad P_1 = 76 + 0.02(76)\left(1 - \frac{76}{420}\right) = 77.245$

5 $\quad t_2 = 2 \quad P_2 = 77.245 + 0.02(77.245)\left(1 - \frac{77.245}{420}\right) = 78.506$

$t_3 = 3 \quad P_3 = 78.506 + 0.02(78.506)\left(1 - \frac{78.506}{420}\right) = 79.782$

Euler's Formula $P_{n+1} =$ $P_n + 0.02P_n\left(1 - P_n/420\right)$

$P(1) \simeq$ 77.245 M $\quad P(2) \simeq$ 78.506 M $\quad P(3) \simeq$ 79.782 M

d. The solution to the logistic growth model is

$$P(t) = \frac{420}{1 + 4.526\,e^{-0.02t}}.$$

What is the percent error between this model and the population in 1950? At what population does the logistic growth model predict the U. S. will level off?

4

$P(50) =$ 157.597M $\quad$ Percent Error = 4.16%

Limiting Population of U. S. = 420 M

e. Another model (Nonautonomous Malthusian growth model) that we studied is given by

$$\frac{dP}{dt} = (0.0183 - 0.000181\,t)P, \qquad P(0) = 76.0.$$

Solve this differential equation. Use this model to predict the population in 1950 and determine the percent error from this model? What is the maximum population for the U. S. predicted by this model and when does it predict this will occur?

$$\int \frac{dP}{P} = \int (0.0183 - 0.000181t)\,dt$$

$$\ln(P(t)) = 0.0183t - \frac{0.000181t^2}{2} + c$$

$$P(t) = 76.0\, e^{0.0183t - 0.0000905t^2}$$

10

$P(t) =$ $76e^{0.0183t - 0.0000905t^2}$

$P(50) =$ 151.33M Percent Error = 0.022%

$t_{max} =$ 101.1 $P(t_{max}) =$ 191.67M

3 f. Give a reason why the Nonautonomous Malthusian growth model is better than the other models, then give a reason why the Logisitc growth model is better than the other models. (Use your calculations above to justify your reasoning.)

Nonautonomous Malthusian growth model has the smallest error at 1950, but maximum population is below current population.

Logistic model is reasonally close and levels at a carrying capacity, common for most animals, but uncertain for humans.