Give all answers to at least **4 significant figures**.

1. When a monoculture of an organism is grown in a limited (but renewed) medium, then the population of that organism often follows the logistic growth model. Below is a table with population data for two species of yeast growing for 90 hours in monocultures of a limited medium. The populations are given in 1000/cc.

t (hr)	Species X	Species Y
0	0.71	0.62
10	1.82	1.13
20	3.77	1.91
30	6.44	3.03
40	8.25	3.96
50	8.91	4.87
60	9.32	5.55
70	9.46	6.04
80	9.52	6.25
90	9.48	6.18

a. The logistic growth model for species X is given by

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{M}\right), \qquad X(0) = X_0.$$

Similarly, the logistic growth model for species Y is given by

$$\frac{dY}{dt} = sY\left(1 - \frac{Y}{N}\right), \qquad Y(0) = Y_0.$$

Give the general solution to each of these models, then find the best parameters X_0, Y_0, r, s, M, and N that fit the data above. (A reasonable first guess for the parameters r and s is 0.1 for both of them.) Also, give the sum of square errors between the data and the model with the best fitting parameters for each of the species. What is the limiting population or carrying capacity for each of these monocultures?

b. When these two species are combined in a single experiment with limited medium, they compete for the available nutrient. The competition model satisfies the system of differential equations given by:

$$\begin{aligned} \frac{dX}{dt} &= rX\left(1 - \frac{X}{M}\right) - bXY = F(X,Y), \\ \frac{dY}{dt} &= sY\left(1 - \frac{Y}{N}\right) - cXY = G(X,Y), \end{aligned}$$

where the parameters r, s, M, and N are the same as parameters found for the monocultures. Suppose that experiments find that the species interaction terms are $b = 0.025$ and $c = 0.014$. Use this information to find the four possible equilibria for this competition model. (Equilibria are found by solving $dX/dt = 0$ and $dY/dt = 0$.)

c. There is no exact solution to this system of differential equations. However, the solution of this system can be approximated using Euler's method with the following formula:

$$
\begin{aligned}
X_{n+1} &= X_n + h\,F(X_n, Y_n), \\
Y_{n+1} &= Y_n + h\,G(X_n, Y_n),
\end{aligned}
$$

where h is the stepsize. Let $X(0) = X_0 = 0.5$ and $Y(0) = Y_0 = 0.5$ with $h = 1$. Simulate this model for 200 hours. Give the approximate populations of X and Y at times $t = 25$, 50, 100, and 200 hours. Does this model exhibit competitive exclusion or coexistence of the species? If this model exhibits competitive exclusion, then determine the maximum population of the species that goes extinct and when that occurs. If there is coexistence, then determine how long until both species are within 90% of the coexistence equilibrium. Describe what happens in this culture for a long period of time. (Use your information from the equilibria calculated in Part b.)

2. In lecture we noted that damped oscillators arise in a number of biological applications. A damped oscillator is derived from a mass-spring problem with resistance and satisfies a second order differential equation. We briefly examined these differential equations at the beginning of our section on differential equations.

a. Consider the damped oscillator given by the differential equation:

$$
\frac{d^2y}{dt^2} + 0.4\frac{dy}{dt} + 9.04y = 0, \qquad y(0) = 1, \quad \frac{dy(0)}{dt} = 2,
$$

where $\frac{d^2y}{dt^2}$ is the second derivative of the unknown function $y(t)$. Solve this problem using Maple. (See Special Maple Help Page.) Find the first two relative minima and maxima for $t \in [0, 6]$. Also, determine the absolute minimum and absolute maximum for $t \in [0, 6]$.

b. One very important physiological system that has evolved to use damped oscillators is our hearing. The basilar membrane (inside the inner ear) is a structure that extends along the cochlea with 20,000-30,000 reed-like fibers that stretch across the width of the cochlea, beginning with short stiff structures and becoming longer and less stiff. These fibers are tuned to specific frequencies and respond like a forced damped oscillator. They vibrate and stimulate the hair cells, which produce nerve signals, allowing us to hear particular frequencies of sound. Sound enters the ear as a compression wave with the wavelength determining the sound. The reed-like fibers of the basilar membrane act like damped oscillators that are being forced by a particular frequency.

A simplified model for the displacement of the reed-like fibers (RLF) ($y(t)$) is given by the differential equation:

$$
\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F_0 \sin(at), \qquad y(0) = 0, \quad \frac{dy(0)}{dt} = 0,
$$

where t is in milliseconds and the forcing function $F_0 \sin(at)$ represents the incoming sound. Let the damping coefficient $c = 0.2$ (which in general would vary with the thickness of the structure) and assume that the amplitude of the sound is given by $F_0 = 5$. There are two parameters remaining in this model, k, which represents the characteristics of the particular RLFs, and a, which gives the frequency of the incoming sound waves.

Let $k = 9.01$ and find the value of the solution for $a = 2.85$ at times $t = 2$ and 20. Repeat this for $a = 3.0$ at times $t = 2$ and 20.

c. In this part of the study we examine how the RLFs respond to sounds. Start with $k = 9.01$ to test how this RLF responds to the six test sounds with $a = 2.85$, 3.0, 3.15, 3.3, 3.45, and 3.6 (or 454, 477, 501, 525, 549, and 573 Hz, respectively). Create the solutions to the differential equation with each of the six test sounds. Use Maple to plot the solutions for $t \in [0, 50]$. Use the plots to estimate the maximum response of the RLF to each of the six test sounds to within 0.5, *i.e.*, state that the maximum response is between say 4 and 4.5 (simply using the graphs). If there is a threshold value of $y \geq 6$ for at least 10 oscillations to stimulate the nerve, then determine which of the test sounds above result in a nerve signal occurring.

d. Repeat Part c with $k = 12.01$.