Give all answers to at least **4 significant figures**.

1. Type 1 or juvenile diabetes is a very dangerous disease caused by an autoimmune response to the β-cells in the pancreas. The earlier the diagnosis of the disease, the better the chances of controlling it with insulin and helping the subject live longer. One simple test for diagnosis is the glucose tolerance test (GTT), where the subject ingests a large amount of glucose (1.75 mg/kg body wt) then has his or her blood monitored for about 6 hours following the glucose administration. Ackerman *et al* [1] created a simple mathematical model for glucose and insulin regulation that is given by the equation

$$G(t) = G_0 + Ae^{-\alpha t}\cos(\omega t + \delta).$$

where $G(t)$ is the blood glucose level (in mg/dl of blood). The GTT allows the fitting the parameters G_0, A, α, ω, and δ.

a. Consider the data below from a couple of patients. Find the best fitting parameters, G_0, A, α, ω, and δ in the equation for $G(t)$. Write the sum of square errors between the data and the model.

t (min)	$G_1(t)$ mg/dl	$G_2(t)$ mg/dl
0	75	105
0.5	160	190
0.75	180	205
1	155	225
1.5	95	200
2	75	185
2.5	65	110
3	80	100
4	85	85
5	80	90

b. Graph the best fitting solution for each of these patients. Using the model, find the absolute maximum and absolute minimum for each of these patients giving both the time, t, and glucose level, G, at the extrema in the 5 hour range of the data.

c. Experimental testing of this model has shown that the parameter α varied from subject to subject, so was not a good predictor of diabetes. However, the parameter $\omega_0 = \sqrt{\omega^2 + \alpha^2}$ was quite robust and proved a good indicator of diabetes. In particular, healthy individuals satisfied $2\pi/\omega_0 < 4$, while the reverse inequality indicated diabetes. Use this information to determine if the data above come from a normal or a diabetic patient.

2. Pollution in fresh water is a major ecological problem for today's society. One of the central laws responsible for cleaning the water in the U. S. is the Clean Water Act of 1972, which was enacted in no small part due to the horrible condition of Lake Erie in the middle of the last century.

a. We begin this problem using a simple model for pollution in Lake Erie based on data from

Rainey [2]. A basic mathematical model for the concentration of a pollutant, $c(t)$, in a well-mixed lake is given by the differential equation:

$$\frac{dc}{dt} = \frac{kr}{V} - \frac{r}{V}c, \qquad c(0) = c_0,$$

where k is the concentration of pollutant entering the lake at a rate r (which is also the rate water leaves), V is the volume of the lake, and c_0 is the initial concentration of the pollutant in the lake. The volume of Lake Erie is $V = 460$ km^3, and its flow rate is $r = 175$ km^3/yr. Solve this differential equation, including the unknown constants k and c_0.

b. Assume that a relatively new pesticide is detected in the lake, and is monitored over a period of time. Below is a table of the concentrations of the pesticide found in Lake Erie.

t (yr)	0	1	2	3	4	5
c (ppb)	2.1	2.8	3.5	3.9	4.3	4.5

Use the solution from Part a and Excel's Solver to find the best fitting constants k and c_0 (least squares best fit) for this model based on the data in the table above. Also, give the sum of square errors for these best constants.

c. If the pesticide entering the lake is stopped completely, $k = 0$, then determine how long until the concentration of the pollutant drops to half the amount seen in Year 5 from the table. Find the concentrations 5 and 10 years after the pesticide is stopped in Year 5.

d. Unfortunately, pesticides remain in the environment and are slowly washed out. Assume that a law is enacted to ban this particular pesticide in Year 5. If it is found that the residual pesticide enters in an exponentially declining manner, then a new model for the concentration of pollutant in Lake Erie is

$$\frac{dc}{dt} = \frac{4.5r\,e^{-0.15t}}{V} - \frac{r}{V}c, \qquad c(0) = c_0,$$

where the constants r and V are from Part a and $c_0 = 4.5$. Use Euler's method with a stepsize of $h = 0.25$(yr) to estimate the solution of this differential equation on the time interval $t \in [0, 10]$. Give the concentrations at times $t = 1, 2, 3, 5, 7,$ and 10 years after the pesticide is stopped. Compare the answers at $t = 5$ and 10 to the ones in Part c.

e. Use Maple's **dsolve** to find the actual solution to the differential equation above. Determine how long it takes for the level of the pollutant to drop to half the amount found in Year 5. Use the solution to determine the percent error between the Euler solution in Part d and the actual solution.

[1] Ackerman, E., Rosevear, J. W., and McGuckin, W. F. (1964). A mathematical model of the glucose tolerance test, *Phys. Med. Biol.*, **9**, 202-213.

[2] R. H. Rainey (1967), Natural displacement of pollution from the Great Lakes, *Science*, **155**, 1242-1243.