1. Evaluate the following integrals:

a. $\displaystyle\int \left(6\cos(3x) - \frac{2}{x^3}\right) dx,$

b. $\displaystyle\int \left(4x + e^{-3x}\right) dx.$

c. $\displaystyle\int \left(\frac{3}{x^2} + 3\,\cos(3x-2)\right) dx,$

d. $\displaystyle\int \left(2x\,e^{-x^2} - 4\,x\right) dx.$

e. $\displaystyle\int \left(4e^{-2x} + \frac{3}{\sqrt{x}}\right) dx,$

f. $\displaystyle\int \left(5x^2 - 1\right)^2 dx.$

g. $\displaystyle\int \left(x^2 + 4\,x - 5\right)^3 (x+2) dx,$

h. $\displaystyle\int \left(\frac{7}{x} + 8\sin^3(4x)\cos(4x)\right) dx.$

2. Solve the following initial value problems:

a. $\dfrac{dy}{dt} = 1 + e^{-t}, \quad y(0) = 3,$

b. $\dfrac{dy}{dt} = 2 - \dfrac{4}{t}, \quad y(1) = 5,$

c. $\dfrac{dy}{dt} = \dfrac{3t^2}{2y}, \quad y(0) = 4,$

d. $\dfrac{dy}{dt} = 2 - 0.02\,y, \quad y(0) = 5,$

e. $\dfrac{dy}{dt} = \dfrac{2\,ty}{t^2+1}, \quad y(0) = 3,$

f. $\dfrac{dy}{dt} = (2 - 0.2t)y, \quad y(0) = 10,$

g. $\dfrac{dy}{dt} = 4 - 2\,\sin(2(t-3)), \quad y(3) = 5,$

h. $\dfrac{dy}{dt} = e^{t-y}, \quad y(0) = 6,$

3. A ball is tossed vertically into the air (with only gravity acting on it) with an initial velocity of 48 ft/sec from a 160 ft platform ($h'(0) = v(0) = 48$ and $h(0) = 160$). Assume that its height, $h(t)$ (in ft), above the ground t seconds after it is thrown satisfies the differential equation $h'' = -32$ ft/sec^2.

a. What is the maximum height of the ball and when does it occur?

b. When does the ball hit the ground and what is its velocity then?

c. Sketch a graph of the height of the ball vs. t for $t \geq 0$.

4. A kangeroo can leap vertically 8 ft. Determine equations describing the velocity and height of the kangeroo as functions of time using this assumption on the maximum height it can achieve, that is find $v(t)$ and $h(t)$ including numerical values of all constants in these formulae. How long is the kangeroo in the air and what is the animal's initial upward velocity? (Use the acceleration due to gravity as 32 ft/sec^2.)

5. It has been shown that the radial spread of a disease in an orchard satisfies the differential equation

$$\frac{dT}{dt} = k\sqrt{T},$$

where T is the number of diseased trees and t is in years. Suppose that initially there is a single diseased tree ($T(0) = 1$) and that 4 years later $T(4) = 25$. Solve this differential equation. Find the value of k, then determine how many trees are infected after 10 years.

6. a. A mathematical model for the growth of tumors is given by Gompertz differential equation. The number of cancer cells (in thousands), $N(t)$, satisfies the differential equation

$$\frac{dN}{dt} = -0.1N \ln\left(\frac{N}{2000}\right), \qquad N(0) = 10.$$

Solve this differential equation.

b. Find what happens to the tumor for very large time.

7. a. A colony of bacteria grows according to the Malthusian growth model

$$\frac{dB}{dt} = 0.01\,B, \qquad B(0) = 1000,$$

where t is in min. Solve this differential equation and determine how long it takes for this population to double.

b. Because it takes a short time to adjust to the new medium, a better model is given by

$$\frac{dB}{dt} = 0.01(1 - e^{-t})\,B, \qquad B(0) = 1000.$$

Solve this differential equation.

c. Compare the populations predicted at $t = 5$ and 60 min.

8. a. Consider the Malthusian growth model for a particular animal that has recently colonized some region

$$\frac{dP}{dt} = 0.2\,P, \qquad P(0) = 100,$$

where t is in years. Solve this differential equation and determine how long it takes for this population to double.

b. Because of habitat encroachment, this animal is losing its range for expansion. This results in a growth rate that is time dependent. Suppose that the population satisfies the modified Malthusian growth model

$$\frac{dP}{dt} = (0.2 - 0.02\,t)\,P, \qquad P(0) = 100.$$

Solve this differential equation.

c. Find the maximum of this population and what year this occurs. Also, determine when the population returns to 100. Sketch a graph for this population.

9. a. Consider the growth of a population of cells in a declining medium. If the population growth depends on the absorption of the medium through the cell surface and the medium is decaying exponentially, then a differential equation for this population is given by

$$\frac{dP}{dt} = 0.3e^{-0.01t}P^{2/3}, \qquad P(0) = 1000,$$

where the initial population is 1000 and t is in hours. Solve this differential equation.

b. Find how long it takes for this population to double. What happens to this population for very large time (*i.e.*, find any horizontal asymptotes)? Sketch a graph for this population.

10. a. An initially clean pond that contains 10,000 m^3 of water maintains a constant volume. The stream flows in at a rate of 200 m^3/day with 10 $\mu g/m^3$ of phosphate (from fertilizer) in the stream. The pond is well-mixed, and a stream flows out at the same rate. Write a differential equation that describes the concentration of phosphate $c(t)$ in the lake, then solve this equation.

b. Algae grows well on phosphate. The rate of growth of algae is proportional to the concentration of phosphate and the population of algae $A(t)$ to the 2/3 power,

$$\frac{dA}{dt} = 0.05\, c(t) A^{2/3}, \qquad A(0) = 1000.$$

Find the population of algae at any time t.

11. There is a tremendous controversy about swordish. It is a very popular sportfish and top ocean predator. There is a boycott of this type of fish because the average harvest size has dropped well below the size of sexual maturity. It is also one of the earliest fish that was recognized to have a buildup of mercury. Sadly, the current swordfish that are being served are too small and young to even have a chance to buildup toxic levels of mercury with the average catch size being only 40 kg, which is less than 3 years old.

a. Swordfish can get extremely large, exceeding 1000 kg. However, it grows slowly (and if not overfished can live a long time), so the weight of a swordfish satisfies the following differential equation:

$$\frac{dw}{dt} = 0.015(1000 - w), \qquad w(0) = 0,$$

where w is the weight in kg and t is the time in years. Solve this differential equation. Use the results to determine how long it takes to produce a mature 70 kg swordfish.

b. The mercury (Hg) accumulates as the swordfish grows and is not removed. Assume that the intake of Hg is proportional to the weight of the swordfish, so satisfies the differential equation

$$\frac{dH}{dt} = kw(t), \qquad H(0) = 0,$$

with $k = 0.01$ (mg of Hg/kg-yr) and H being the mg of Hg in a swordfish. Solve this differential equation. Find the amount of Hg in swordfish that are 3 and 20 years old.

c. If the Hg is uniformly spread in the swordfish, then the concentration of Hg, $c(t)$ (in μg/g), would be given by the formula

$$c(t) = H(t)/w(t).$$

Find the weight of swordfish, $w(t)$, and concentration of Hg, $c(t)$, at times $t = 3$ and 20 years. (Note that officials get concerned when Hg level reaches 0.1 μg/g.)

12. Studies of Lake Apopka in Florida show that the alligators there have been exposed to high levels of various estrogen-simulating pesticides, like DDT (or its breakdown DDE), dieldrin, and toxaphenes. Apparently, this exposure has resulted in dramatic decrease in the size of the male

alligator penises, which in turn finally spurred our Congress into action. (They were unconcerned when it was shown to have adverse effects on female animal populations.) The levels of these estrogenic pesticides in this lake far exceed the safe levels established by the EPA. The effect on penis development is clearly related to the amount of exposure of the embryo, which in turn reflects the amount in the body of the female alligators.

a. The growth (weight) of a female alligator can be approximated by the following differential equation:

$$\frac{dw}{dt} = 0.2(80 - w), \qquad w(0) = 0,$$

where w is the weight in kg and t is the time in years. Solve this differential equation. Use the results to determine how long it takes to produce a mature 40 kg alligator.

b. The pesticides accumulate (especially in the fatty tissues) as the alligator grows and is not removed. Assume that the intake of pesticides is proportional to the weight of the alligator, so satisfies the differential equation

$$\frac{dP}{dt} = kw(t), \qquad P(0) = 0,$$

with $k = 600$ (μg/kg-yr) and P being the μg of pesticides in an alligator at Lake Apopka. Solve this differential equation. Find the amount of pesticide in an alligator that is 5 years old.

c. The concentration, $c(t)$ (in μg/g or ppm), is found by computing

$$c(t) = \frac{P(t)}{1000w(t)}.$$

Find the concentration in a 5 year old alligator. (Note that officials get concerned when pesticide levels reach 0.1 ppm in an animal.)

13. A new pesticide is introduced to a particular region, where a stream becomes contaminated with the pesticide and flows into a 10^6 m^3 lake. The stream flows at a rate of 4000 m^3/day with a concentration of 15 ng/m^3.

a. Assuming that the lake is well-mixed and maintains a constant volume, then the differential equation describing the concentration of this new pesticide in the lake is given by the differential equation:

$$\frac{dc}{dt} = 0.004(15 - c), \qquad \text{with} \qquad c(0) = 0.$$

Solve this differential equation and find the limiting concentration in the lake.

b. More realistically, the flow of the stream is seasonal and fluctuates over the year by about 40%. The lake still maintains an almost constant volume, but a better model for the concentration of the pesticide in the lake is given by:

$$\frac{dc}{dt} = -0.001(4 - \cos(0.0172t))(c - 15), \qquad \text{with} \qquad c(0) = 0.$$

Solve this differential equation. Be sure to show your steps for this calculation, including the separation of variables and the solution of each of the integrals.

14. This problem examines pollution entering a well-mixed lake. The concentration, $c(t)$, of the pollutant in the lake with volume V maintained by a river flowing in and out at a rate f satisfies the differential equation

$$\frac{dc(t)}{dt} = \frac{f}{V}(Q(t) - c(t)),$$

where $Q(t)$ is the concentration of pollutant entering the lake from the river.

a. Assume $c(0) = 0$, $V = 10,000\ \text{m}^3$, $f = 200\ \text{m}^3/\text{day}$, and $Q(t) = 10$ ppb. Solve this differential equation and find how long it takes for the lake to have a concentration of 2 ppb of pollutant.

b. Suppose that the level of pollutant is increasing linearly in the river, so $Q(t) = 10 + 0.1t$. Use Euler's method with $h = 1$ and 2 steps to approximate the solution at $t = 2$.

15. a. The decay of a particular fruit with total mass, M, satisfies the following differential equation:

$$\frac{dM}{dt} = -k\, M^{3/4}, \qquad M(0) = 16\ \text{g},$$

where t is in days. It is found that after 10 days only 1 g remains of the fruit, so $M(10) = 1$. Solve this differential equation. Find the value of k. Determine when the fruit completely vanishes ($M(t_f) = 0$).

b. A special culture of bacteria is added to the decaying fruit, and it is found that the decaying fruit satisfies the differential equation:

$$\frac{dM}{dt} = -0.8\, e^{-0.02t}\, M^{3/4}, \qquad M(0) = 16\ \text{g},$$

Solve this differential equation. Find the length of time for this fruit to completely vanish.

16. a. A population study is conducted on a new colony of invasive insects. The study finds that initially there are 60 of the insects in 1 m^2, $P(0) = 60$. Two weeks later a survey finds 80 of the insects in 1 m^2, $P(2) = 80$. Assume that this insect population is growing according to a Malthusian growth law:

$$\frac{dP}{dt} = r\, P,$$

where t is in weeks. Solve this differential equation, find the growth constant r, and determine how long it takes for the total population to double.

b. It is found that a predator is adapting to the new invasive insect and is learning to control this pest. A survey after four weeks finds the population has only increased to 90, $P(4) = 90$. The result is a declining growth rate and a better model for the population is given by the differential equation:

$$\frac{dP}{dt} = (a - bt)P.$$

Solve this differential equation. Use the data at $t = 0$, 2, and 4 weeks to find the constants a and b. Determine the time for this population to reach its maximum and what the maximum population is predicted to be.