1. Differentiate the following: (Do **NOT** simplify!)

a. $f(x) = \sin(3x - 5) + \ln(\cos(3x))$

b. $g(x) = \dfrac{4}{\cos(x^2 + 2)} - (x^2 - \sin^3(x^2))^4$

c. $h(x) = \dfrac{x^4 + e^{-2x}}{x^3 + \cos(4x)} + e^{-x}\cos(2x)$

d. $k(x) = (x^2 - 5)^3 \cos(x^3) - e^{\sin(2x)}$

2. Solve the following differential equations:

a. $\dfrac{dy}{dt} = -0.2y, \quad y(0) = 8.$

b. $\dfrac{dx}{dt} = 3 - 0.1x, \quad x(0) = 4.$

c. $\dfrac{dw}{dt} = 0.02w + 4, \quad w(0) = 2.$

d. $\dfrac{dh}{dx} = -\dfrac{h}{5}, \quad h(0) = 50.$

e. $\dfrac{dy}{dt} = 2 + \dfrac{y}{3}, \quad y(0) = 2.$

f. $\dfrac{dz}{dt} = 0.3z, \quad z(4) = 10.$

3. Consider the function:

$$f(t) = \frac{\sin(2t)}{\cos(2t)}.$$

a. Find the derivative of $f(t)$. Evaluate $f'(0)$. Is the derivative positive, negative, or both for $t \geq 0$?

b. For $t \in [0, 2\pi]$, find all zeroes of $f(t)$. Also, determine where the function is undefined (zeroes of the denominator). The zeroes in the denominator give the location of the vertical asymptotes.

c. Use the information from Parts a and b to sketch a graph of $f(t)$ for $t \in [0, 2\pi]$.

4. A damped spring-mass system has a solution of the form

$$y(t) = 2e^{-2t}\sin(2t),$$

where $y(t)$ measures the distance in centimeters from the equilibrium position and t is in seconds.

a. Find the velocity of the mass by computing the derivative, $v(t) = y'(t)$.

b. Find the time $t > 0$ and the position y when the mass is at a maximum. Also, determine the first time after $t = 0$ when $y(t) = 0$ again.

5. The displacement of specific fibers on the basilar membrane stimulates the hair cells, which send a signal to the auditory part of the brain, indicating a particular wavelength of sound has been heard. For a given tone, assume that the basilar fiber vibrates according to the equation:

$$z(t) = 15e^{-t/2}\sin(t/2),$$

where $z(t)$ measures the distance in microns from the rest position of the fiber and t is in milliseconds.

a. Determine all times when $z(t) = 0$ for $t \in [0, 4\pi]$.

b. Find the velocity of the basilar fiber by computing the derivative, $v(t) = z'(t)$.

c. Find the times $t \in [0, 4\pi]$ when the basilar fiber is at a maximum and a minimum. Give the values of z at these extrema. Sketch a graph of $z(t)$ for $t \in [0, 4\pi]$.

6. In lab we considered a model for the length of day in San Diego. The effect is dramatically more pronounced when you go to Alaska. In Anchorage, Alaska, the longest day is June 20 at 19 hr 22 min or 1162 min. The shortest day is December 21 at 5 hr 27 min or 327 min. Consider a model for the length of the day in minutes, $L(t)$, as a function of the date, t, using the sine function as follows

$$L(t) = \alpha + \beta \sin(\omega(t - \phi)),$$

where the constants α, β, ω, and ϕ are to be determined below (and assuming that January 1 is $t = 0$).

a. Assume that June 20 is given by day 170 with length of 1162 min. With the information that the shortest day is 327 min and a year is 365 days, find the constants α, β, ω, and ϕ. Write the function $L(t)$ and find the length of Ground Hog day (February 2 or Day 32) in Anchorage.

b. Differentiate $L(t)$ to find the rate of change in the length of day and write this formula. Find the date when this rate of change is increasing the most, and what is the rate of change per day at that date?

7 a. Initially, there are 1000 bacteria in a particular culture. If this culture is growing according to the Malthusian growth law and two hours later there are 3000 bacteria, then write a differential equation describing the growth of these bacteria, $B(t)$, solve the equation, and find the doubling time for this culture.

b. Suppose that a single mutant cell, $M(t)$, enters the culture (at $t = 0$) and satisfies the differential equation

$$M'(t) = 0.7M(t).$$

Solve this differential equation, find the doubling time of this mutant bacteria, then determine how long until the populations $B(t)$ and $M(t)$ are equal.

8 a. The population of Japan was 116.8 million in 1980, while in 1990, it was 123.5 million. Let 1980 be $t = 0$ (giving the initial population) and assume that Japan's population satisfies the Malthusian growth law given by

$$\frac{dJ}{dt} = rJ, \qquad J(0) = J_0,$$

where r is the growth rate, J_0 is the initial population, and t is time in years. Solve this growth model for Japan, giving the growth rate r from these data to **4 significant figures**. How long does it take for Japan's population to double according to this model?

b. The population of Bangladesh was 88.1 million in 1980, while in 1990, it was 110.1 million. Again, let 1980 be $t = 0$ and assume that Bangladesh's population satisfies a Malthusian growth law. Write a differential equation describing the population of Bangladesh, $B(t)$, solve this equation, and use the model to predict Bangladesh's population in 2000. (Give the growth rate to **4 significant figures**.)

c. Find when the population of Japan is equal to the population of Bangladesh according to the models above.

9. The Malthusian growth model with immigration is given by the initial value problem:

$$\frac{dP}{dt} = rP + m, \qquad P(0) = P_0,$$

where r is the growth rate, m is the immigration rate, and P_0 is the initial population. Which of the following possible solutions satisfies this equation? Show it is a solution.

(i). $P(t) = (P_0 - m)\, e^{rt} + \frac{m}{r}$
(ii). $P(t) = (P_0 + m)\, e^{rt} - m$
(iii). $P(t) = \left(P_0 + \frac{m}{r}\right) e^{rt} - \frac{m}{r}$

10. Radioactive iodine, ^{131}I, is used to treat a number of thyroid problems, including hyperthyroidism, Grave's disease, and thyroid cancer. The first two of these problems require only 3-12 mCi (millicuries) of ^{131}I, so can be treated using an outpatient procedure. However, the cancer treatment often uses doses of 30 mCi and requires a special isolation room where everything must be disposed of as radioactive waste, including urine, feces, bedsheets, and eating utensils, since the ^{131}I is contained in all the bodily fluids (though concentrated in the thyroid).

a. Radioactive iodine, ^{131}I, has a half-life of 8 days and satisfies the differential equation

$$R' = -kR, \quad R(0) = R_0.$$

Assume that $R_0 = 30$ mCi, then find the decay constant k (to 3 significant figures). If the patient is isolated for 3 days, then how many millicuries of ^{131}I remain either in the patient or with the waste products collected.

b. How long does it take for the original 30 mCi dose to decay to only 5 mCi? (Note that most of the ^{131}I will have passed out through the fluids and not be in the body by this time.)

11. The population of the United States was about 50.2 million in 1880 and 62.9 million in 1890. Let 1880 be represented by $P(0)$ and assume that its population is growing according to the Malthusian growth law,

$$\frac{dP(t)}{dt} = rP(t),$$

where t is in years.

a. Use the data above to find the growth rate r, then solve the differential equation above. Determine how long until the U. S. population doubled from its 1880 level according to this model.

b. Predict the population in the year 1900. The actual population was about 76.0 million. What is the error between the model and the actual census data?

12. a. White lead is a pigment found in oil paints and can be used to detect art forgeries. In the absence of radium-226, lead-210 undergoes standard radioactive decay,

$$P' = -kP.$$

Suppose that a sample from a painting has 10 disintegrations per minute in 1970 and then shows 8.5 disintegrations per minute in 1975. Find the half-life of lead-210 and give the value of k.

b. When there are impurities caused by radium-226 (which has a very long half-life), the differential equation for radioactive decay is modified to

$$P' = -kP + r,$$

where $r = 0.25$ is source input from the radium-226 and k is from Part a. Solve this differential equation and determine the limit of P (disintegrations per minute of lead-210) as $t \to \infty$.

13. You are attending a conference, and the talks are going past the coffee break time. You really need a cup of tea (not liking coffee) to keep awake for the next set of talks. The refreshments are in a room that has a constant temperature of 21°C, and you find that the hot water is only 85°C. Five minutes later, the hot water is only 81°C.

a. Assume that the container of water satisfies Newton's law of cooling. ($H' = -k(H - T_e)$, where T_e is the environmental temperature.) If it was placed out when the talks were supposed to end with boiling water (water at 100°C), then how many minutes beyond the scheduled time did the talks go? (Hint: If $H(t)$ is the temperature, then use $H(0) = 85$ and $H(5) = 81$ to find the cooling constant k in Newton's law of cooling, then find when $H(t) = 100$.)

b. If tea needs water that is at least 93°C to give you enough caffeine for the next set of talks, then how long after the scheduled end of the talks can you wait?

14. a. An initially clean lake ($c(0) = 0$) concentrates pollution from an incoming stream because of evaporative loss of water of 200 m^3/day. The well-mixed lake has a stream flowing in at a rate of $f_1 = 2200$ m^3/day with a pesticide concentration of of $Q = 10$ ppb. The lake maintains a constant volume of $V = 10^6$ m^3 by having a stream leaving with a flow of $f_2 = 2000$ m^3/day. You are given that the differential equation describing the concentration of pesticide in the lake is given by

$$c' = \frac{1}{V}(f_1Q - f_2c).$$

Solve this differential equation.

b. Determine how long until the lake has a concentration of 5 ppb of pesticide. Also, find the limiting concentration of pesticide. Sketch a graph of the solution.

15. This problem examines pollution entering a well-mixed lake. The concentration, $c(t)$, of the pollutant in the lake with volume V maintained by a river flowing in and out at a rate f satisfies the differential equation

$$\frac{dc(t)}{dt} = \frac{f}{V}(Q(t) - c(t)),$$

where $Q(t)$ is the concentration of pollutant entering the lake from the river.

a. Assume $c(0) = 0$, $V = 10,000$ m^3, $f = 200$ m^3/day, and $Q(t) = 10$ ppb. Solve this differential equation and find how long it takes for the lake to have a concentration of 2 ppb of pollutant.

b. Suppose that the level of pollutant is increasing linearly in the river, so $Q(t) = 10 + 0.1t$. Use Euler's method with $h = 1$ and 2 steps to approximate the solution at $t = 2$.

16. Consider the differential equation:

$$\frac{dy}{dt} = t\,(2-y), \qquad y(0) = 4.$$

a. Use Euler's method with $h = 0.25$ to approximate the solution at $t = 1$.

b. The solution to this differential equation is one of the following choices:

(i). $y(t) = 2 + 2e^{-t}$,

(ii). $y(t) = t^2 + 4$,

(iii). $y(t) = 2 + 2e^{-\frac{t^2}{2}}$.

Evaluate $y(1)$, then determine the percent error at $t = 1$ from the approximation in Part a using Euler's method.

17. Consider the differential equation:

$$\frac{dy}{dt} = y + 2, \qquad y(0) = 3.$$

a. Use Euler's method with $h = 0.25$ to approximate the solution at $t = 1$.

b. Find the solution to this differential equation and evaluate $y(1)$. Determine the percent error at $t = 1$.

18. a. A radioactive substance satisfies the differential equation

$$\frac{dR}{dt} = -0.05R, \qquad R(0) = 10.$$

Find the solution of this differential equation and determine the half-life of this radioactive substance.

b. Radioactive elements are often the products of the decay of another radioactive element. A differential equation describing this situation is given by the following:

$$\frac{dR}{dt} = -0.05R + 0.2e^{-0.01t}, \qquad R(0) = 10.$$

Use Euler's method with a stepsize of $h = 1$ to find the approximate solution at $t = 3$.

c. The solution to the problem in Part b is one of the following choices:

(i). $R(t) = 8e^{-0.05t} + 2e^{-0.01t}$ (ii). $R(t) = 5e^{-0.05t} + 5e^{-0.01t}$

Select the correct solution and verify your answer. Use this answer to determine the percent error between your Euler's approximate solution and the actual solution at $t = 3$.

19. An older woman is quite ill, and her daughter finds that she has been running a temperature of 39°C. Over the night, the woman passes away in her sleep, and the daughter discovers her death at 7 AM. At this time the body is found to be 35°C. Two hours later the body temperature is

33.5°C. The woman's bedroom maintained a temperature of 25°C. If the body satisfies Newton's law of cooling,

$$\frac{dH(t)}{dt} = -k(H(t) - T_e),$$

where T_e is the temperature of the bedroom, t is in hours, H is the temperature in °C, and k is the coefficient of heat transfer to be determined (to **4 significant figures**) for this woman. Determine when the woman died (using normal time, hours and minutes).

20. a. A new experimental drug is being tested for its efficacy on treatment of tumors. The patient (who has none of this drug in her body to begin with) has 10 liters of blood. The drug enters the blood intravenously at a rate of 1 liter/day at a concentration of 0.2 μg/liter. The body mixes this drug well throughout the circulatory system, then it filters out the drug through the kidneys at a rate proportional to the concentration of the drug in the blood with a flow rate of 1 liter/day. Set up the differential equation that describes the concentration, $c(t)$, of this new drug and solve it.

b. Find when the drug concentration reaches a concentration of 0.1 μg/liter in the body, so that the tumor responds.

c. Assume the same scenario as given above for the drug entering and leaving the patient's body. If it is found that the body metabolizes (removes) 0.05 μg/day of the drug, then write the new differential equation that describes the concentration of this drug in the body. What is the limiting concentration of the drug in the body for this model?