Some notes on row reduction

Let's say we want to find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 0 & -5 \\ 3 & -1 & 4 \end{pmatrix}$$

using row reduction. Here is the procedure:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -4 & 0 & -5 & 0 & 1 & 0 \\ 3 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\textstyle R_3 + (3/4)R_2 \to R_3]{\substack{R_1 + R_2/4 \to R_1 \\ -R_2/4 \to R_2}} \left(\begin{array}{ccc|ccc} 0 & 2 & \frac{7}{4} & 1 & \frac{1}{4} & 0 \\ 1 & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & -1 & \frac{1}{4} & 0 & \frac{3}{4} & 1 \end{array}\right)$$

$$\xrightarrow{\substack{R_1 + 2R_3 \to R_1 \\ -R_3 \to R_3}} \left(\begin{array}{ccc|ccc} 0 & 0 & \frac{9}{4} & 1 & \frac{7}{4} & 2 \\ 1 & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{4} & 0 & -\frac{3}{4} & -1 \end{array}\right)$$

$$\xrightarrow{\substack{(4/9)R_1 \to R_1 \\ R_2 - (5/9)R_1 \to R_2 \\ R_3 + (1/9)R_1 \to R_3}} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & \frac{4}{9} & \frac{7}{9} & \frac{8}{9} \\ 1 & 0 & 0 & -\frac{5}{9} & -\frac{11}{9} & -\frac{10}{9} \\ 0 & 1 & 0 & \frac{1}{9} & -\frac{5}{9} & -\frac{7}{9} \end{array}\right)$$

Hence

$$A^{-1} = \frac{1}{9}\begin{pmatrix} -5 & -11 & -10 \\ 1 & -5 & -7 \\ 4 & 7 & 8 \end{pmatrix}.$$

Now you can easily check that this is correct by multiplying out

$$A(9A^{-1}) = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 0 & -5 \\ 3 & -1 & 4 \end{pmatrix}\begin{pmatrix} -5 & -11 & -10 \\ 1 & -5 & -7 \\ 4 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = 9I.$$

If you got this right, great. But what if you made a mistake along the way? Before I show you how to find a mistake, let me remind you of how row-reduction works to find the inverse of a matrix. All along, we are trying to solve the linear system of equations:

$$AX = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 0 & -5 \\ 3 & -1 & 4 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

for X by doing simultaneous row operations on A and I. You can think of this as three linear systems of equations

$$A\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad A\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad A\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

combined together for efficiency. Let's write out the first of these in all its detail:

$$\begin{aligned} x_{11} + 2x_{21} + 3x_{31} &= 1 \\ -4x_{11} - 5x_{31} &= 0 \\ 3x_{11} - x_{21} + 4x_{31} &= 0 \end{aligned}$$

The row operations we do on A and I correspond to multiplying one of these equations by a nonzero number and adding (subtracting) it to (from) another to eliminate the unknowns one-by-one. Once we have arrived at the solution, we could go back and substitute it into any equation along the way, and it should satisfy that equation. In matrix terms, we could take the left half of any of the augmented matrices along the way, multiply it by our solution for A^{-1} and we should get the right half of the same augmented matrix. If you do this to the very first augmented matrix, whose left side is A and right side is I, you are just verifying that $AA^{-1} = I$.

Now, let's say we botched adding the fractions when computing the middle entry in the right matrix in the last step. We were supposed to subtract $(5/9)(7/4)$ from $-1/4$, but we accidentally added them and got

$$\frac{5}{9}\frac{7}{4} - \frac{1}{4} = \frac{35}{36} - \frac{9}{36} = \frac{26}{36} = \frac{13}{18}.$$

Our (incorrect) computation looks like this now

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -4 & 0 & -5 & 0 & 1 & 0 \\ 3 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\substack{R_1 + R_2/4 \to R_1 \\ -R_2/4 \to R_2 \\ R_3 + (3/4)R_2 \to R_3}} \left(\begin{array}{ccc|ccc} 0 & 2 & \frac{7}{4} & 1 & \frac{1}{4} & 0 \\ 1 & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & -1 & \frac{1}{4} & 0 & \frac{3}{4} & 1 \end{array}\right)$$

$$\xrightarrow{\substack{R_1 + 2R_3 \to R_1 \\ -R_3 \to R_3}} \left(\begin{array}{ccc|ccc} 0 & 0 & \frac{9}{4} & 1 & \frac{7}{4} & 2 \\ 1 & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{4} & 0 & -\frac{3}{4} & -1 \end{array}\right) \xrightarrow{\substack{(4/9)R_1 \to R_1 \\ R_2 - (5/9)R_1 \to R_2 \\ R_3 + (1/9)R_1 \to R_3}} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & \frac{4}{9} & \frac{7}{9} & \frac{8}{9} \\ 1 & 0 & 0 & -\frac{5}{9} & \frac{13}{18} & -\frac{10}{9} \\ 0 & 1 & 0 & \frac{1}{9} & -\frac{5}{9} & -\frac{7}{9} \end{array}\right)$$

And our alleged inverse matrix looks like:

$$A^{-1} = \begin{pmatrix} -\frac{5}{9} & \frac{13}{18} & -\frac{10}{9} \\ \frac{1}{9} & -\frac{5}{9} & -\frac{7}{9} \\ \frac{4}{9} & \frac{7}{9} & \frac{8}{9} \end{pmatrix}.$$

That 13/18 catches our attention. It looks like something might have gone wrong. Let's verify. When we multiply the first row of A with the second column of our alleged A^{-1}, we get

$$1\,\frac{13}{18} - 2\,\frac{5}{9} + 3\,\frac{7}{9} = \frac{13}{18} - \frac{10}{9} + \frac{7}{3} = \frac{13 - 20 + 42}{18} = \frac{35}{18}.$$

This should of course be 0. How can we find where we made the mistake?

If you had an equation, tried to solve it, and came up with an incorrect solution, how would you try to find the mistake? You would go back in your computation, one step at a time, and you would substitute your incorrect solution into that step to see whether it satisfies the equation. You keep backtracking until you find an equation that is no longer satisfied by your incorrect solution. It is between this step and the next that you made the mistake.

What we have here is a system of equations. So we can follow the same strategy. We know we have a mistake in the second column, so let's try to plug in the second column first into the last augmented matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{13}{18} \\ -\frac{5}{9} \\ \frac{7}{9} \end{pmatrix} = \begin{pmatrix} \frac{7}{9} \\ \frac{13}{18} \\ -\frac{5}{9} \end{pmatrix}$$

So far so good, this is indeed the second column on the right side of the last augmented matrix. Let's try the third augmented matrix:

$$\begin{pmatrix} 0 & 0 & \frac{9}{4} \\ 1 & 0 & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{13}{18} \\ -\frac{5}{9} \\ \frac{7}{9} \end{pmatrix} = \begin{pmatrix} \frac{7}{4} \\ \frac{61}{18} \\ * \end{pmatrix} \neq \begin{pmatrix} \frac{7}{4} \\ -\frac{1}{4} \\ -\frac{3}{4} \end{pmatrix}$$

We've found it, we didn't even have to compute the last entry. So something went wrong in computing the middle entry between the third and the fourth augmented matrices. Let's redo that computation. What was it again? It was $R_2 - (5/9)R_1 \to R_2$, so for the 2nd entry of the 2nd row:

$$-\frac{1}{4} - \frac{5}{9}\frac{7}{4} = -\frac{9}{36} - \frac{35}{36} = -\frac{44}{36} = -\frac{11}{9}.$$

Looks much better. Now we have

$$A^{-1} = \frac{1}{9} \begin{pmatrix} -5 & -11 & -10 \\ 1 & -5 & -7 \\ 4 & 7 & 8 \end{pmatrix}.$$

Let us check this

$$A(9A^{-1}) = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 0 & -5 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} -5 & -11 & -10 \\ 1 & -5 & -7 \\ 4 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = 9I.$$

It's time to rejoice.

All of this takes a lot of writing to explain, but in fact doesn't take very much to do. Still, if you find it too much trouble to look for a mistake like this, you may prefer to just start from scratch and redo the entire row-reduction. It doesn't take that long. When you do, you may want to choose different rows for pivots, otherwise there is a chance you'll make the same mistake as before.