Math 5C Midterm Solutions

White version of exam

Lecture 2, S'03, May 3, 2003

1. (a) $\begin{pmatrix} -1 & 3 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 10 & 4 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -16 & 11 \\ -100 & 25 \end{pmatrix}.$

   (b) $\begin{pmatrix} 2 & -1 & 3 \\ 2 & 0 & 5 \\ 3 & 1 & 4 \end{pmatrix}^{-1} = \frac{1}{-11}\begin{pmatrix} -5 & 7 & 2 \\ 7 & -1 & -5 \\ -5 & -4 & 2 \end{pmatrix}^T = \frac{1}{11}\begin{pmatrix} 5 & -7 & 5 \\ -7 & 1 & 4 \\ -2 & 5 & -2 \end{pmatrix}.$

2. (a) Find the Taylor series for $f(x) = 1/x^2$ at $x = 3$ by differentiating f a few times and observing a pattern.

$$\begin{aligned} f(x) &= \frac{1}{x^2} \implies f(3) = \frac{1}{3^2} \\ f'(x) &= \frac{-2}{x^3} \implies f'(3) = \frac{-2}{3^3} \\ f''(x) &= \frac{(-2)(-3)}{x^4} \implies f''(3) = \frac{(-2)(-3)}{3^4} \\ f^{(3)}(x) &= \frac{(-2)(-3)(-4)}{x^5} \implies f^{(3)}(3) = \frac{(-2)(-3)(-4)}{3^5} \\ &\vdots \\ f^{(n)}(x) &= \frac{(-1)^n(n+1)!}{x^{n+2}} \implies f^{(n)}(3) = \frac{(-1)^n(n+1)!}{3^{n+2}} \end{aligned}$$

$$f(x) \approx \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!}{n!3^{n+2}}(x-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{3^{n+2}}(x-3)^n.$$

   (b) What is the radius of convergence? What is the interval of convergence?

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+2)}{3^{n+3}}(x-3)^{n+1}}{\frac{(-1)^n(n+1)}{3^{n+2}}(x-3)^n} \right| = \lim_{n\to\infty} \left| \frac{n+2}{n+1}\,\frac{x-3}{3} \right| = \left| \frac{x-3}{3} \right| < 1$$

$$\implies |x-3| < 3 \implies 0 < x < 6$$

   So the radius of convergence is 3.

3. (a) Prove that if A and B are orthogonal matrices, so is $A^{-1}B$.

   We know $A^T = A^{-1}$ and $B^T = B^{-1}$. We want to show $(A^{-1}B)^T = (A^{-1}B)^{-1}$.

$$(A^{-1}B)^T = (A^TB)^T = B^T(A^T)^T = B^TA = B^{-1}A = (A^{-1}B)^{-1} \quad \checkmark$$

   (b) Prove that if A is an orthogonal matrix, so is A^T.

$$A^T = A^{-1} \implies A^TA = I \implies (A^T)^{-1} = A = (A^T)^T.$$

   (c) Extra credit: Use the result of part (b) to conclude that a matrix A is orthogonal if and only if its <u>row</u> vectors are orthonormal.

By (b), we have that if A is orthogonal then A^T is orthogonal. Use this for A^T to show if A^T is orthogonal then $(A^T)^T = A$ is orthogonal. So A is orthogonal if and only if A^T is orthogonal. But A^T is orthogonal if and only if its column vectors are orthonormal. The column vectors of A^T are the row vectors of A. So we can conclude A is orthogonal if and only if its row vectors are orthonormal.

4. Use the method of Frobenius to find solutions of the DE

$$2xy'' + y' - 2y = 0.$$

(a) First notice that $x_0 = 0$ is a regular singularity. So assume $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Substitute this into the DE and determine the indicial equation and the recursion relation.

$$\begin{aligned}
0 &= 2x\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} - 2\sum_{n=0}^{\infty} a_n x^{n+r} \\
&= x^r\left[\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n-1} + \sum_{n=0}^{\infty} a_n(n+r)x^{n-1} - \sum_{n=1}^{\infty} 2a_{n-1}x^{n-1}\right] \\
&= \Big[2a_0 r(r-1) + a_0 r\Big]x^{r-1} + x^r\sum_{n=1}^{\infty}\Big[a_n(n+r)\big(2(n+r-1)+1\big) - 2a_{n-1}\Big]x^{n-1}
\end{aligned}$$

We need

$$\begin{aligned}
2a_0 r(r-1) + a_0 r &= 0 \\
a_0 r(2r-2+1) &= 0
\end{aligned}$$

As a_0 is assumed to be nonzero, the indicial equation is

$$r(2r-1) = 0.$$

The recursion relation:

$$\begin{aligned}
0 &= a_n(n+r)(2n+2r-1) - 2a_{n-1} \\
a_n &= \frac{2a_{n-1}}{(n+r)(2n+2r-1)} = \frac{4a_{n-1}}{(2n+2r)(2n+2r-1)}.
\end{aligned}$$

(b) Find two linearly independent generalized power series solutions. Check your solutions by substituting them back into the DE.

The roots of the indicial equation are $r = 0$ and $r = 1/2$. So let's look at these two cases:

Case $r = 0$:

$$\begin{aligned}
a_n &= \frac{4a_{n-1}}{(2n)(2n-1)} = \frac{4^2 a_{n-2}}{(2n)(2n-1)(2n-2)(2n-3)} = \cdots \\
&= \frac{4^n a_0}{(2n)(2n-1)\cdots 1} = \frac{4^n a_0}{(2n)!}
\end{aligned}$$

We want one particular solution out of this, so let's choose $a_0 = 1$:

$$y_1(x) = \sum_{n=0}^{\infty} \frac{4^n}{(2n)!}x^n = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!}x^n.$$

Case $r = 1/2$:

$$\begin{aligned}
a_n &= \frac{4a_{n-1}}{(2n+1)(2n)} = \frac{4^2 a_{n-2}}{(2n+1)(2n)(2n-1)(2n-2)} = \cdots \\
&= \frac{4^n a_0}{(2n+1)(2n)\cdots 2} = \frac{4^n a_0}{(2n+1)!}
\end{aligned}$$

We want one particular solution out of this, so let's choose $a_0 = 1$:

$$y_2(x) = \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)!} x^{n+1/2} = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n+1)!} x^{n+1/2}.$$

Checking:

$$
\begin{aligned}
&2xy_1'' + y_1' - 2y_1 = \\
&= 2x \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} n(n-1)x^{n-2} + \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} n x^{n-1} - 2\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} x^n \\
&= \sum_{n=1}^{\infty} \frac{2^{2n+1}}{(2n)!} n(n-1)x^{n-1} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} n x^{n-1} - \sum_{n=1}^{\infty} \frac{2^{2n-1}}{(2n-2)!} x^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} \underbrace{\left(2n(n-1) + n - \frac{(2n)(2n-1)}{2}\right)}_{=2n^2-2n+n-2n^2+n=0} x^{n-1} = 0 \qquad \checkmark
\end{aligned}
$$

$$
\begin{aligned}
&2xy_2'' + y_2' - 2y_2 = \\
&= 2x \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n+1)!} \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) x^{n-3/2} \\
&\quad + \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n+1)!} \left(n+\frac{1}{2}\right) x^{n-1/2} - 2\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n+1)!} x^{n+1/2} \\
&= \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} \left(n^2-\frac{1}{4}\right) x^{n-1/2} + \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n+1)!} \left(n+\frac{1}{2}\right) x^{n-1/2} - \sum_{n=1}^{\infty} \frac{2^{2n-1}}{(2n-1)!} x^{n-1/2} \\
&= \frac{2^0}{1!} \underbrace{\left[2\frac{-1}{4} + \frac{1}{2}\right]}_{=0} x^{-1/2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n+1)!} \underbrace{\left[2n^2 - \frac{1}{2} + n + \frac{1}{2} - \frac{(2n+1)(2n)}{2}\right]}_{=2n^2+n-n(2n+1)=0} x^{n-1/2} = 0 \qquad \checkmark
\end{aligned}
$$

(c) What does the Theorem of Frobenius say about the radii of convergence of these solutions?

That their radii of convergence are at least as large as the minimum of radii of convergence of $xP(x) = x(1/2x) = 1/2$ and $x^2Q(x) = x^2(1/x) = x$. As the latter are both analytic, their radii of convergence are infinite, so the radii of convergence of the solutions are also infinite.

(d) Extra credit: Can you figure out from the power series what these functions are? (If you can, you can use them to check your solutions.)

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} x^n = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} \sqrt{x}^{2n} = \sum_{n=0}^{\infty} \frac{(2\sqrt{x})^{2n}}{(2n)!} = \cosh(2\sqrt{x}) \\
y_2(x) &= \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n+1)!} x^{n+1/2} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} \sqrt{x}^{2n+1} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(2\sqrt{x})^{2n+1}}{(2n+1)!} = \frac{1}{2}\sinh(2\sqrt{x}).
\end{aligned}
$$

5. Let

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -3 & 1 \end{pmatrix}.$$

(a) Find an orthogonal matrix B such that $B^{-1}AB$ is diagonal. Verify that B is orthogonal by checking that its columns are orthonormal.

First, let's find the eigenvalues of A:

$0 = (\lambda - 5)\big((\lambda-1)^2 - 9\big) = (\lambda-5)(\lambda-1+3)(\lambda-1-3) \implies \lambda = 5, 4, -2.$

Now we'll find eigenvectors of unit length. Note that we don't have to worry about orthogonality in this case because eigenvalues corresponding to the different eigenvalues are automatically orthogonal to each other.

$$\lambda_1 = 5: \qquad (A-5I)\vec{x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & -3 \\ 0 & -3 & -4 \end{pmatrix} \vec{x_1} = \vec{0} \implies \vec{x_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 4: \qquad (A-4I)\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{pmatrix} \vec{x_2} = \vec{0} \implies \vec{x_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\lambda_3 = -2: \qquad (A+2I)\vec{x} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \vec{x_3} = \vec{0} \implies \vec{x_3} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

So

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 1, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1,$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0. \quad \checkmark$$

(b) Find the general solution of the linear system

$$\frac{d^2\vec{x}}{dt^2} = A\vec{x}.$$

Substitute $\vec{x} = B\vec{y}$:

$$B\frac{d^2\vec{y}}{dt^2} = AB\vec{y} \implies \frac{d^2\vec{y}}{dt^2} = B^{-1}AB\vec{y} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix} \vec{y}.$$

So

$$\begin{pmatrix} y_1'' \\ y_2'' \\ y_3'' \end{pmatrix} = \begin{pmatrix} 5y_1 \\ 4y_2 \\ -2y_3 \end{pmatrix} \implies \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_1 e^{\sqrt{5}t} + b_1 e^{-\sqrt{5}t} \\ a_2 e^{2t} + b_2 e^{-2t} \\ a_3 \sin(\sqrt{2}t) + b_3 \cos(\sqrt{2}t) \end{pmatrix}.$$

After choosing new arbitrary constants $A_1 = a_1$, $A_2 = a_2/\sqrt{2}$, $A_3 = a_3/\sqrt{2}$ and similarly $B_1 = b_1$, $B_2 = b_2/\sqrt{2}$, $B_3 = b_3/\sqrt{2}$,

$$\vec{x} = B\vec{y} = \begin{pmatrix} A_1 e^{\sqrt{5}t} + B_1 e^{-\sqrt{5}t} \\ A_2 e^{2t} + B_2 e^{-2t} + A_3 \sin(\sqrt{2}t) + B_3 \cos(\sqrt{2}t) \\ A_2 e^{2t} + B_2 e^{-2t} - A_3 \sin(\sqrt{2}t) - B_3 \cos(\sqrt{2}t) \end{pmatrix}.$$