MATH 579 FINAL EXAM SOLUTIONS

Dec 12, 2007

1. (10 pts) Let $k, n \in \mathbb{Z}^{\geq 0}$ such that $k < n$. Prove the following identity, called the Hexagon Identity.

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}.$$

$$\begin{aligned}\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} &= \frac{(n-1)!}{(k-1)!(n-1-(k-1))!}\,\frac{n!}{(k+1)!(n-(k+1))!}\,\frac{(n+1)!}{k!(n+1-k)!}\\ &= \frac{(n-1)!\,n!\,(n+1)!}{(k-1)!(n-k)!(k+1)!(n-k-1)!k!(n+1-k)!}\\ &= \frac{(n-1)!}{k!(n-k-1)!}\,\frac{n!}{(k-1)!(n-k+1))!}\,\frac{(n+1)!}{(k+1)!(n-k)!}\\ &= \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}\end{aligned}$$

2. (15 pts) Use the Principle of Inclusion and Exclusion to count the prime numbers up to 168.

First note that $\sqrt{168} < \sqrt{169} = 13$. So any composite number up to 168 must have a prime factor less than 13. Such primes are $2, 3, 5, 7, 11$. Let

$$S = \{x \in \mathbb{Z}^+ \mid x \leq 168\}$$

and

$$S_p = \{x \in \mathbb{Z}^+ \mid x \leq 168, p|x\}.$$

Since every p-th number in S is divisible by p, $|S_p| = \lfloor 168/p \rfloor$. Similarly, $|S_p \cap S_q| = \lfloor 168/(pq) \rfloor$ for different primes p, q, etc. We can now use the Principle of Inclusion and Exclusion to count the integers between 1 and 168 that are not divisible by $2, 3, 5, 7, 11$:

$$\begin{aligned}|S| - \sum_{p\in\{2,3,5,7,11\}} |S_p| + \sum_{\substack{p,q\in\{2,3,5,7,11\}\\ p<q}} |S_p \cap S_q| - \sum_{\substack{p,q,r\in\{2,3,5,7,11\}\\ p<q<r}} |S_p \cap S_q \cap S_r|&\\ = 168 - \sum_{p\in\{2,3,5,7,11\}} \left\lfloor \frac{168}{p} \right\rfloor + \sum_{\substack{p,q\in\{2,3,5,7,11\}\\ p<q}} \left\lfloor \frac{168}{pq} \right\rfloor - \sum_{\substack{p,q,r\in\{2,3,5,7,11\}\\ p<q<r}} \left\lfloor \frac{168}{pqr} \right\rfloor&\\ = 168 - (84 + 56 + 33 + 24 + 15) +&\\ (28 + 16 + 12 + 7 + 11 + 8 + 5 + 4 + 3 + 2) -&\\ (5 + 4 + 2 + 2 + 1 + 1 + 1 + 1)&\\ = 35&\end{aligned}$$

But we want to count primes. So we should include 2,3,5,7,11 and exclude 1. Hence the number of primes between 1 and 168 is actually $35 + 5 - 1 = 39$.

3. (15 pts) A ransom note demands \$10000 in unmarked fifty and hundred-dollar bills. There are two kinds of bills in both denominations: the old-fashioned bills and the new anticounterfeit bills. Use a generating function argument to compute the number of ways to combine such bills to make \$10000.

For the sake of simpler notation, we will count in units of \$50. Let f_k be the number of ways to make \$$50k$ using old \$50 bills only, F_k the number of ways to make \$$50k$ using old and new \$50 bills, h_k the number of ways to make \$$50k$ using old and new \$50 bills and old \$100 bills, H_k the number of ways to make \$$50k$ using old and new \$50 bills and old and new \$100 bills.

First construct the four generating functions for f_k, F_k, h_k, and H_k:

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} f_k x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x} \\
F(x) &= \sum_{k=0}^{\infty} F_k x^k = (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots) = \frac{1}{(1-x)^2} \\
h(x) &= \sum_{k=0}^{\infty} h_k x^k = (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots) \\
&= \frac{1}{(1-x)^2(1-x^2)} \\
H(x) &= \sum_{k=0}^{\infty} H_k x^k \\
&= (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^2 + x^4 + \cdots) \\
&= \frac{1}{(1-x)^2(1-x^2)^2}
\end{aligned}$$

where each x represents \$50. We are looking for H_{200}. We immediately notice that $f_k = 1$ for all k. Now,

$$\begin{aligned}
f(x) = (1-x)F(x) &= (1-x)\sum_{k=0}^{\infty} F_k x^k \\
&= \sum_{k=0}^{\infty} F_k x^k - \sum_{k=0}^{\infty} F_k x^{k+1} \\
&= \sum_{k=0}^{\infty} F_k x^k - \sum_{k=1}^{\infty} F_{k-1} x^k \\
&= F_0 + \sum_{k=1}^{\infty} (F_k - F_{k-1}) x^k
\end{aligned}$$

This gives the relation

$$f_k = \begin{cases} F_k & \text{if } k = 0 \\ F_k - F_{k-1} & \text{if } k \geq 1. \end{cases}$$

Solving for F_k, we get

$$F_k = \begin{cases} f_k & \text{if } k = 0 \\ f_k + F_{k-1} & \text{if } k \geq 1. \end{cases}$$

Similarly,

$$\begin{aligned}
F(x) = (1-x^2)h(x) &= (1-x^2)\sum_{k=0}^{\infty} h_k x^k \\
&= \sum_{k=0}^{\infty} h_k x^k - \sum_{k=0}^{\infty} h_k x^{k+2} \\
&= \sum_{k=0}^{\infty} h_k x^k - \sum_{k=2}^{\infty} h_{k-2} x^k \\
&= h_0 + h_1 x + \sum_{k=2}^{\infty} (h_k - h_{k-2}) x^k.
\end{aligned}$$

This gives the relation

$$F_k = \begin{cases} h_k & \text{if } 0 \le k \le 1 \\ h_k - h_{k-2} & \text{if } k \ge 2. \end{cases}$$

Solving for h_k, we get

$$h_k = \begin{cases} F_k & \text{if } 0 \le k \le 1 \\ F_k + h_{k-2} & \text{if } k \ge 2. \end{cases}$$

Finally,

$$\begin{aligned}
h(x) = (1-x^2)H(x) &= (1-x^2)\sum_{k=0}^{\infty} H_k x^k \\
&= \sum_{k=0}^{\infty} H_k x^k - \sum_{k=0}^{\infty} H_k x^{k+2} \\
&= \sum_{k=0}^{\infty} H_k x^k - \sum_{k=2}^{\infty} H_{k-2} x^k \\
&= H_0 + H_1 x + \sum_{k=2}^{\infty} (H_k - H_{k-2}) x^k.
\end{aligned}$$

This is just like the previous relation, so it will yield

$$H_k = \begin{cases} h_k & \text{if } 0 \le k \le 1 \\ h_k + H_{k-2} & \text{if } k \ge 2. \end{cases}$$

We want to find H_{200}. Observe that

$$\begin{aligned}
H_{200} &= h_{200} + H_{198} \\
&= h_{200} + h_{198} + H_{196} \\
&\vdots \\
&= h_{200} + h_{198} + h_{196} + \cdots h_2 + h_0
\end{aligned}$$

We only have h_{2n} in this sum, so it is enough to figure out what those are. Using the recursion for h_k, we find

$$\begin{aligned} h_{2n} &= F_{2n} + h_{2n-2} \\ &= F_{2n} + F_{2n-2} + h_{2n-4} \\ &\vdots \\ &= F_{2n} + F_{2n-2} + F_{2n-4} + \cdots + F_2 + F_0 \end{aligned}$$

As for F_k, we have

$$\begin{aligned} F_k &= f_k + F_{k-1} \\ F_k &= f_k + f_{k-1} + F_{k-2} \\ &\vdots \\ F_k &= f_k + f_{k-1} + f_{k-2} + \cdots + f_1 + f_0 = k+1 \end{aligned}$$

since $f_k = 1$ for all k. Quick reality check: this makes sense as the only choice we have at this point is how many of the k bills are old. The rest must be new. Out of k bills, $0, 1, \ldots, k$ could be old. This yields $k+1$ different choices.

We can now use this to find

$$\begin{aligned} h_{2n} &= F_{2n} + F_{2n-2} + F_{2n-4} + \cdots + F_2 + F_0 \\ &= (2n+1) + (2n-1) + \cdots + 3 + 1 = (n+1)^2 \end{aligned}$$

where we used the well-known identity

$$1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2.$$

(You can easily prove this by induction.) Finally

$$\begin{aligned} H_{200} &= h_{200} + h_{198} + h_{196} + \cdots h_2 + h_0 \\ &= 101^2 + 100^2 + 99^2 + \cdots + 2^2 + 1^2 \\ &= \frac{101 \cdot 102 \cdot 203}{6} = 348551 \end{aligned}$$

where we used the other well-known identity

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

This is also easy to prove by induction.

4. (10 pts) Suppose you have marbles of 5 different colors: red, blue, green, yellow, and black. There are many marbles of each. You put them all in a big bag, then you reach in and take out k of them. Use an appropriate generating function to find how many different outcomes are possible.

   First note that the function which generates all possible outcomes is

$$\begin{aligned} F = {} & (1 + x + x^2 + \cdots)(1 + y + y^2 + \cdots) \\ & (1 + z + z^2 + \cdots)(1 + u + u^2 + \cdots)(1 + v + v^2 + \cdots). \end{aligned}$$

The number we are looking for the is the number of terms of total degree k in this formal power series. Since we don't care about each individual outcome, only how many of them there are, we will set $x = y = z = u = v$. This will turn all terms of total degree k into x^k. Now we just need to determine the coefficient of x^k in

$$F = (1 + x + x^2 + \cdots)^5 = \frac{1}{(1-x)^5}.$$

We know from calculus that the coefficient of x^k in this power series is

$$\frac{F^{(k)}(0)}{k!}.$$

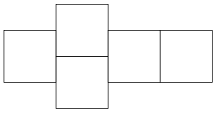
Let's look at the first few derivatives:

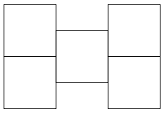
$$\begin{aligned}
F_1 &= \frac{F^{(1)}(0)}{1!} = \frac{-5}{(1-x)^6}(-1)\bigg|_{x=0} = 5 \\
F_2 &= \frac{F^{(2)}(0)}{1!} = \frac{1}{2!}\,\frac{(-5)(-6)}{(1-x)^7}(-1)^2\bigg|_{x=0} = \frac{5 \cdot 6}{2!} \\
F_3 &= \frac{F^{(3)}(0)}{1!} = \frac{1}{3!}\,\frac{(-5)(-6)(-7)}{(1-x)^8}(-1)^3\bigg|_{x=0} = \frac{5 \cdot 6 \cdot 7}{3!}
\end{aligned}$$

You see the pattern:

$$F_k = \frac{5 \cdot 6 \cdot 7 \cdots (5 + k - 1)}{k!} = \binom{k+4}{k}.$$

5. (10 pts) In the children's game of hopscotch, the player hops across an array of squares drawn on the ground. Each position has either one or two squares, requiring the player to land on one foot, or on both feet. There are two examples of hopscotch games on the left.





   (a) Give a recursive formula for the number of different hopscotch games with exactly n squares.

   One way to construct a hopscotch game with n squares is to take any hopscotch game with $n-1$ squares and add one square to the end. Another way to construct a hopscotch game with n squares is to take any hopscotch game with $n-2$ squares and add two adjacent squares to the end. Actually, these are the only ways to build a hopscotch game with n squares because any such game must end with either one or two squares. Notice that if we start with two different hopscotch games with $n-1$ or $n-2$ squares and add one or two squares to the end as needed, the resulting hopscotch games of n squares will also be different. So the recursive formula for $h(n)$, the number of different hopscotch games of n squares, is $h(n) = h(n-1) + h(n-2)$. To start the recursion, observe that there is only one hopscotch game with 1 square. We can also say that there is exactly one hopscotch game with 0 squares, namely the empty hopscotch array. So the complete formula is

$$\begin{aligned}
h(0) &= h(1) = 1 \\
h(n) &= h(n-1) + h(n-2) && \text{for } n \geq 2.
\end{aligned}$$

   (b) Use your formula to find the number of different hopscotch games with 6 squares.

$$\begin{aligned} h(0) &= h(1) = 1 \\ h(2) &= h(1) + h(0) = 2 \\ h(3) &= h(2) + h(1) = 3 \\ h(4) &= h(3) + h(2) = 5 \\ h(5) &= h(4) + h(3) = 8 \\ h(6) &= h(5) + h(4) = 13 \end{aligned}$$

So there are 13 different hopscotch games with 6 squares.

(c) Do you recognize these numbers? What are they called?

They are the obviously the Fibonacci numbers.

6. (20 pts) Let S be a nonempty set.

(a) State the definition of a permutation on S.

A permutation of S is a one-to-one correspondence (or bijection) $\sigma : S \to S$.

(b) Let σ and ϕ be permutations on S. Prove that $\sigma \circ \phi$ is also a permutation on S.

Since $\sigma : S \to S$ and $\phi : S \to S$, it is clear that $\sigma \circ \phi$ is $S \to S$. We need to show $\sigma \circ \phi$ is one-to-one (injective) and onto (surjective).
First, let $x \neq y$ in S. Then $\phi(x) \neq \phi(y)$ because ϕ is one-to-one, and $\sigma(\phi(x)) \neq \sigma(\phi(y))$ because σ is also one-to-one. Hence $\sigma \circ \phi$ is one-to-one.
Now let $z \in S$. Since σ is onto, there exists a $y \in S$ such that $\sigma(y) = z$. But ϕ is also onto, so there is an $x \in S$ such that $\phi(x) = y$. Hence $\sigma \circ \phi(x) = z$. This can be done for any $z \in S$, which shows that $\sigma \circ \phi$ is onto.

(c) Consider the permutation given in two-row notation by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 1 & 2 & 5 \end{pmatrix}.$$

Write this in disjoint cycle notation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 1 & 2 & 5 \end{pmatrix} = (1\,4)(2\,6\,5).$$

(d) Let $\sigma = (153)(64)$ in S_7. Write σ in two-row notation.

$$(153)(64) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 1 & 6 & 3 & 4 \end{pmatrix}.$$

(e) Compute in disjoint cycle notation

$$(1\,4)(2\,1\,5)(3\,7\,4).$$

$$(1\,4)(2\,1\,5)(3\,7\,4) = (1\,5\,2\,4\,3\,7).$$

(f) Compute in disjoint cycle notation

$$((2\,7\,1)(3\,6)(5\,4))^{-1}.$$

$$((2\,7\,1)(3\,6)(5\,4))^{-1} = (4\,5)(6\,3)(1\,7\,2).$$

7. (20 pts) **Extra credit problem.** Let $N \in \mathbb{Z}^+$ and

$$f(x) = \prod_{1 \le i < j \le n} (j - i).$$

Let $\sigma \in S_n$. Define

$$f_\sigma(x) = \prod_{1 \le i < j \le n} \big(\sigma(j) - \sigma(i)\big).$$

(a) Prove that for any $\sigma \in S_n$,

$$f_\sigma(x) = \pm f(x).$$

Let $i < j$. Then $\sigma(i) \neq \sigma(j)$ since σ is one-to-one. Hence either $\sigma(i) < \sigma(j)$ or $\sigma(j) < \sigma(i)$. In the first case, $(\sigma(j) - \sigma(i))$ is a factor of f. In the second case, $(\sigma(i) - \sigma(j)) = -(\sigma(j) - \sigma(i))$ is a factor of f. It is easy to see that because σ is one-to-one, different factors of f remain different after action by σ. Notice that for each factor $(j - i)$ of f, either $(\sigma^{-1}(j) - \sigma^{-1}(i))$ or $(\sigma^{-1}(i) - \sigma^{-1}(j))$ is also a factor of f. Hence $(j - i)$ is also a factor in f_σ possibly up to a negative sign. This shows that up to sign, the factors of f and of f_σ are the same. Hence $f_\sigma = \pm f$.

(b) Prove that if σ is a transposition, then $f_\sigma(x) = -f(x)$.

Let $\sigma = (k\,m)$. Without loss of generality, we may assume $k < m$. Let $i \neq k, m$. We have the following cases:

$i < k$: Then $(k - i)$ and $(m - i)$ are both factors of f. Applying σ to f only switches these two factors. Switching factors does not change the value of f.

$m < i$: Then $(i - k)$ and $(i - m)$ are both factors of f. Applying σ to f only switches these two factors. Switching factors does not change the value of f.

$k < i < m$: Then σ changes $(i - k)$ to $(i - m)$ and $(m - i)$ to $(k - i)$. Since $i < m$, $(i - m)$ is not a factor or f, but $(m - i) = -(i - m)$ is. Similarly, $(i - k) = -(k - i)$ is a factor of f. The σ switches the factors $(i - k)$ and $(m - i)$, and introduces negative signs for both. Switching factors does not change the value of f and the two negative signs of course cancel.

These cases cover almost all of the factors of f. There is just one more factor, namely $(m - k)$. Applying σ changes this to $(k - m)$. This is of course $-(m - k)$. So this alters the sign of f, but not its absolute value. This one sign change accounts for the fact that $f_\sigma = -f$.

(c) Conclude that there is no permutation in S_n, which is equal to a product of an even number of transpositions and an odd number of transpositions at the same time.

First, notice that

$$(f_\sigma)_\phi = f_{\phi\sigma}.$$

This is because $\phi\sigma$ is be definition the permutation ϕ composed with the permutation σ and so $\phi\sigma$ acting on the numbers $1, 2, \ldots, n$ is the same as acting with σ followed by ϕ.

Suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ where the σ_i are transpositions. Then

$$f_\sigma = f_{\sigma_1\sigma_2\cdots\sigma_k} = \Big(\big((f_{\sigma_k})_{\sigma_{k-1}}\big)_{\sigma_{k-2}}\cdots\Big)_{\sigma_1} = (-1)^k f$$

as each transposition introduces a negative sign. This shows that if σ is a product of an even number of transpositions, then $f_\sigma = f$ and if σ is a product of an odd number of transpositions, then $f_\sigma = -f$. Obviously, f_σ can't be both f and $-f$ at the same time, so σ cannot be both a product of an even number and an odd number of transpositions.