Math 579 Exam 3 Solutions

Nov 19, 2007

1. (15 pts) The Lucas numbers are defined by

$$\begin{aligned} L_0 &= 2 \\ L_1 &= 1 \\ L_k &= L_{k-1} + L_{k-2} \qquad \text{for } k \geq 2. \end{aligned}$$

Use an appropriate generating function to find a closed formula for L_k. (Hint: your formula will likely be in terms $\phi = (1+\sqrt{5})/2$ and $\hat{\phi} = (1-\sqrt{5})/2$, the roots of the golden ratio equation $x^2 - x - 1$. Note that this means $x^2 + x - 1 = (x+\phi)(x+\hat{\phi})$.)

Let

$$f(x) = \sum_{k=0}^{\infty} L_k x^k.$$

By the recurrence relation above

$$\begin{aligned} f(x) &= L_0 + L_1 x + \sum_{k=2}^{\infty} (L_{k-1} + L_{k-2}) x^k \\ &= 2 + x + \sum_{k=2}^{\infty} L_{k-1} x^k + \sum_{k=2}^{\infty} L_{k-2} x^k \\ &= 2 + x + x \sum_{k=2}^{\infty} L_{k-1} x^{k-1} + x^2 \sum_{k=2}^{\infty} L_{k-2} x^{k-2} \\ &= 2 + x + x \sum_{k=1}^{\infty} L_k x^k + x^2 \sum_{k=0}^{\infty} L_k x^k \\ &= 2 - x + x \left(2 + \sum_{k=1}^{\infty} L_k x^k \right) + x^2 \sum_{k=0}^{\infty} L_k x^k \\ &= 2 - x + x \sum_{k=0}^{\infty} L_k x^k + x^2 \sum_{k=0}^{\infty} L_k x^k \\ &= 2 - x + x f(x) + x^2 f(x) \end{aligned}$$

Hence

$$f(x) = \frac{x-2}{x^2+x-1} = \frac{A}{x+\phi} + \frac{B}{x+\hat{\phi}} \implies x - 2 = A(x+\hat{\phi}) + B(x+\phi).$$

Substitute $x = -\phi$ and $x = -\hat{\phi}$ to find A and B:

$$\begin{aligned} -\phi - 2 = A(-\phi + \hat{\phi}) &\implies A = \frac{2+\phi}{\phi - \hat{\phi}} \\ -\hat{\phi} - 2 = B(-\hat{\phi} + \phi) &\implies B = \frac{2+\hat{\phi}}{\hat{\phi} - \phi}. \end{aligned}$$

Notice that $\phi + \hat{\phi} = \sqrt{5}$. Hence

$$f(x) = \frac{1}{\sqrt{5}} \left(\frac{2+\phi}{x+\phi} - \frac{2+\hat{\phi}}{x+\hat{\phi}} \right).$$

Now use $\phi\hat{\phi} = -1$ to get

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{5}}\left(\frac{(2+\phi)(-\hat{\phi})}{1-\hat{\phi}x} - \frac{(2+\hat{\phi})(-\phi)}{1-\phi x}\right) \\ &= \frac{1}{\sqrt{5}}\left((1-2\hat{\phi})\sum_{k=0}^{\infty}\hat{\phi}^k x^k + (2\phi-1)\sum_{k=0}^{\infty}\phi^k x^k\right) \\ &= \sum_{k=0}^{\infty}\frac{(1-2\hat{\phi})\hat{\phi}^k + (2\phi-1)\phi^k}{\sqrt{5}}x^k \end{aligned}$$

Notice that $1-2\hat{\phi} = 1-(1-\sqrt{5}) = \sqrt{5}$ and $2\phi-1 = (1+\sqrt{5})-1 = \sqrt{5}$. Hence

$$L_k = \hat{\phi}^k + \phi^k.$$

Now for good measure, you can check that

$$L_0 = \hat{\phi}^0 + \phi^0 = 2, \qquad L_1 = \hat{\phi}^1 + \phi^1 = 1.$$

2. (15 pts) Show that the number of ways to make $10m$ cents using only pennies, nickels, and dimes is $(m+1)^2$.

Let p_k be the number of ways to make k cents of change using pennies only, n_k be the number of ways to make k cents of change using pennies and nickels, and d_k be the number of ways to make k cents of change using pennies, nickels, and dimes.

First construct three different generating functions for p_k, n_k, and d_k:

$$\begin{aligned} P(x) &= \sum_{k=0}^{\infty} p_k x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x} \\ N(x) &= \sum_{k=0}^{\infty} n_k x^k = (1+x+x^2+\cdots)(1+x^5+x^{10}+\cdots) \\ &= \frac{1}{(1-x)(1-x^5)} \\ D(x) &= \sum_{k=0}^{\infty} d_k x^k = (1+x+x^2+\cdots)(1+x^5+x^{10}+\cdots)(1+x^{10}+x^{20}\cdots) \\ &= \frac{1}{(1-x)(1-x^5)(1-x^{10})} \end{aligned}$$

It is immediately clear that $p_k = 1$ for all k. Now

$$\begin{aligned} P(x) &= (1-x^5)N(x) = (1-x^5)\sum_{k=0}^{\infty} n_k x^k \\ &= \sum_{k=0}^{\infty} n_k x^k - \sum_{k=0}^{\infty} n_k x^{k+5} = \sum_{k=0}^{\infty} n_k x^k - \sum_{k=5}^{\infty} n_{k-5} x^k \\ &= n_0 + n_1 x + \cdots + n_4 x^4 + \sum_{k=5}^{\infty} n_k x^k - \sum_{k=5}^{\infty} n_{k-5} x^k \\ &= n_0 + n_1 x + \cdots + n_4 x^4 + \sum_{k=5}^{\infty}(n_k - n_{k-5})x^k \end{aligned}$$

Hence

$$p_k = \begin{cases} n_k & \text{if } 0 \leq k \leq 4 \\ n_k - n_{k-5} & \text{if } n \geq 5 \end{cases}.$$

Solving for n_k, we get

$$n_k = \begin{cases} p_k & \text{if } 0 \leq k \leq 4 \\ p_k + n_{k-5} & \text{if } n \geq 5 \end{cases}.$$

Similarly,

$$\begin{aligned} N(x) = (1-x^{10})D(x) &= (1-x^{10})\sum_{k=0}^{\infty} d_k x^k \\ &= \sum_{k=0}^{\infty} d_k x^k - \sum_{k=0}^{\infty} d_k x^{k+10} = \sum_{k=0}^{\infty} d_k x^k - \sum_{k=10}^{\infty} d_{k-10} x^k \\ &= d_0 + d_1 x + \cdots + d_9 x^9 + \sum_{k=10}^{\infty} d_k x^k - \sum_{k=10}^{\infty} d_{k-10} x^k \\ &= d_0 + d_1 x + \cdots + d_9 x^9 + \sum_{k=10}^{\infty} (d_k - d_{k-10}) x^k \end{aligned}$$

Hence

$$n_k = \begin{cases} d_k & \text{if } 0 \leq k \leq 9 \\ d_k - d_{k-10} & \text{if } n \geq 10 \end{cases}.$$

Solving for d_k, we get

$$d_k = \begin{cases} n_k & \text{if } 0 \leq k \leq 9 \\ n_k + d_{k-10} & \text{if } n \geq 10 \end{cases}.$$

Now

$$\begin{aligned} d_{10m} &= n_{10m} + d_{10m-10} = n_{10m} + n_{10m-10} + d_{10m-20} \\ &= \cdots = n_{10m} + n_{10m-10} + \cdots n_0 \\ &= \sum_{j=0}^{m} n_{10j}. \end{aligned}$$

Notice that

$$\begin{aligned} n_{10j} &= p_{10j} + n_{10j-5} = p_{10j} + p_{10j-5} + n_{10j-10} \\ &= \cdots = p_{10j} + p_{10j-5} + \cdots p_0 \\ &= \sum_{i=0}^{2j} p_{5k} = \sum_{i=0}^{2j} 1 = 2j+1 \end{aligned}$$

Therefore

$$\begin{aligned} d_{10m} &= \sum_{j=0}^{m} n_{10j} = \sum_{j=0}^{m} 2j+1 = 2\sum_{j=0}^{m} j + \sum_{j=0}^{m} 1 \\ &= 2\frac{(m+0)(m+1)}{2} + (m+1) = (m+1)(m+1) + (m+1)^2. \end{aligned}$$

3. (10 pts) Let F_k be the Fibonacci numbers defined the usual way:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_k = F_{k-1} + F_{k-2} \qquad \text{for } k \geq 2.$$

Recall that we proved in class the closed formula

$$F_k = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}}$$

where $\phi = (1 + \sqrt{5})/2$ and $\hat{\phi} = (1 - \sqrt{5})/2$ are the roots of the golden ratio equation $x^2 - x - 1$. Prove the identity

$$\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}.$$

Hint: you can do this the easy way or the hard way. The hard way is to use the closed form, the easy way is to use induction.

Let's do this by induction. If $n = 0$, then

$$\sum_{k=0}^{n} F_k^2 = F_0^2 = 0^2 = 0 = F_0 F_1.$$

Suppose the statement is true for n. Now

$$\sum_{k=0}^{n+1} F_k^2 = \sum_{k=0}^{n} F_k^2 + F_{n+1}^2 = F_n F_{n+1} + F_{n+1}^2$$
$$= F_{n+1}(F_n + F_{n+1}) = F_{n+1} F_{n+2}$$

which is exactly what we wanted to prove .

4. (10 pts) Recall the Tower of Hanoi game from class. Let a_k be the number of moves needed to move k disks from one pole to another according to the rules of the game. We showed in class that a_k can be computed using the recurrence relation

$$a_1 = 1$$
$$a_k = 2a_{k-1} + 1 \qquad \text{for } k \geq 2$$

Use a generating function argument to show that $a_k = 2^k - 1$.

We did this in class and the argument is also in your textbook.

5. (15 pts) **Extra credit problem.** Let $n \in \mathbb{Z}^{\geq 0}$. A partition of n into $k \in \mathbb{Z}^+$ parts is a sum

$$n = n_1 + n_2 + \cdots + n_k$$

where $n_1, n_2, \ldots, n_k \in \mathbb{Z}^{\geq 0}$. (This has nothing to do with partitions of sets.) Since addition is commutative, the order of the summands does not matter. For example, $1+2+3$ and $2+2+2$ are different partitions of 6, but $1+2+3$ and $3+2+1$ are the same. In practice, it is customary to list the summands in decreasing order.

Let $P_{n,k}$ be the number of ways to partition n into k parts. For example, $P_{3,3} = 3$ and $P_{6,2} = 4$. Obviously, $P_{0,k} = 1$ and $P_{n,1} = 1$.

(a) For convenience, let us allow n and k to be any integer (possibly negative) and agree that $P_{n,k} = 0$ if $n < 0$ or $k \leq 0$.

Prove the recurrence relation

$$P_{n,k} = P_{n,k-1} + P_{n-k,k}.$$

Let $n = n_1 + n_2 + \cdots + n_k$ be some partition of n into k parts. Let us adopt the convention of listing parts in decreasing order, that is $n_1 \geq n_2 \geq \cdots \geq n_k$.
If $n_k = 0$, then $n = n_1 + n_2 + \cdots + n_{k-1}$ which is also a partition of n into $k-1$ parts. Notice that this process is reversible. The same way we removed the 0 from the end of the sum, we can add it back. In fact, we could start with any partition of n into $k-1$ parts and turn it into a partition of n into k parts with $n_k = 0$ by adding a 0 to the end.
If $n_k > 0$, then $n_i > 0$ for all i. Hence $n_i - 1 \geq 0$ for all i and

$$n - k = (n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1)$$

is a partition of $n-k$ into k parts. Notice that this process is also reversible. The same way we subtracted 1 from each part, we can add 1 back to each part. In fact, starting with any partition of $n-k$ into k parts, we can turn it into a partition of n into k parts, in which all the parts are nonzero.
Therefore each partition of n into k parts can be mapped to either a partition of n into $k-1$ parts or a partition of $n-k$ into k parts via a map that is a one-to-one correspondence. Hence there are as many partititions of n into k parts as there are partitions of n into $k-1$ parts and partitions of $n-k$ into k parts combined. That is

$$P_{n,k} = P_{n,k-1} + P_{n-k,k}.$$

The only problem with the argument above is what happens if either $k-1 = 0$ or $n-k < 0$. In that case, it does not make sense to talk about partitions of n into $k-1$ parts or partitions of $n-k$ into k parts. Since we set $P_{n,k} = 0$ if $n < 0$ or $k \leq 0$, the recurrence relation remains valid even in such cases.

(b) Suppose you have 6 identical hamburgers and 4 identical plates. How many ways are there to distribute the hamburgers on the 4 plates if you can have empty plates too? (Notice that this question is different from the hamburger question on the last exam as the plates are all the same.)

Notice that you are asked to partition 6 hamburgers into 4 parts. So the answer is $P_{6,4}$, which we can compute using the recurrence relation from part (a):

$$\begin{aligned} P_{6,4} &= P_{6,3} + P_{2,4} = P_{6,2} + P_{3,3} + P_{2,3} + P_{-2,4} \\ &= P_{6,1} + P_{4,2} + P_{3,2} + P_{0,3} + P_{2,2} + P_{-1,3} \\ &= 1 + P_{4,1} + P_{2,2} + P_{3,1} + P_{1,2} + 1 + P_{2,1} + P_{0,2} \\ &= 1 + 1 + P_{2,1} + P_{0,2} + 1 + P_{1,1} + P_{-1,2} + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 9. \end{aligned}$$

In case you are curious, here are the 9 partitions (without writing the 0 parts):

$$\begin{aligned} 6 &= 6 = 5 + 1 = 4 + 2 = 4 + 1 + 1 = 3 + 3 = 3 + 2 + 1 \\ &= 3 + 1 + 1 + 1 = 2 + 2 + 2 = 2 + 2 + 1 + 1. \end{aligned}$$

(c) Suppose you have 6 identical hamburgers and as many identical plates as you want. How many ways do you have to distribute the hamburgers on plates if you don't want to put empty plates on the table? (You may use your result from part (b) in answering this question.)

Obviously, we will never need more than 6 plates. So the answer is $P_{6,6}$. We will use the recurrence relation to compute this:

$$\begin{aligned} P_{6,6} &= P_{6,5} + P_{0,6} = P_{6,4} + P_{1,5} + 1 = 9 + P_{1,4} + P_{-4,5} + 1 \\ &= 10 + P_{1,3} + P_{-2,3} = 10 + P_{1,2} + P_{-2,2} = 10 + P_{1,1} + P_{-1,2} = 11. \end{aligned}$$

In case you are curious, here are the 11 partitions (without writing the 0 parts):

$$\begin{aligned} 6 &= 6 = 5+1 = 4+2 = 4+1+1 = 3+3 = 3+2+1 = 3+1+1+1 \\ &= 2+2+2 = 2+2+1+1 = 2+1+1+1+1 = 1+1+1+1+1+1. \end{aligned}$$