Math 579 Exam 1 Solutions

Oct 1, 2007

1. (10 pts) Compute the number of ways to deal a two-pair hand (that is two distinct matching pairs) in single-deck five-card poker.

There are 13 faces in the deck, so there are 13 ways to pick the face for the first pair. There are 4 cards of that face in the deck and we need to pick two out of these 4. There are now 12 faces left to choose the second pair from. Among the 4 cards of this face, we again choose two for the pair. But there isn't really such a thing as the first pair and the second pair. The order of the pairs doesn't matter. Permuting the two pairs does not change the hand. So we need to divide by the number of permutations of the two pairs, which is 2!. Finally, there are 44 cards left whose faces are different from the two pairs we have already chosen. The result is:

$$\frac{1}{2}13\binom{4}{2}12\binom{4}{2}44 = 123552.$$

2. (10 pts) Prove that there are exactly 2^n different subsets of a set of n elements without using the Binomial Theorem.

First, the combinatorial approach. Let S be a set of n elements and let the elements of S be $x_1, x_2, \ldots, x_n$. If $T \subseteq S$, you can describe T by saying whether it contains $x_1, x_2, \ldots, x_n$. Conversely, to form a subset $T \subseteq S$, we decide whether to include x_1 or not, then we decide whether to include x_2 or not, etc. For each element, we choose between two options, and the choices are independent. So altogether we have 2^n possible outcomes for our choices. Each of these outcomes results in a different subset, and each subset of S is realized as one of these outcomes. Hence S has 2^n different subsets.

Alternately, we can use induction. Let $S = \{x_1, x_2, \ldots, x_n\}$. We will induct on n. The base case is $n = 0$. In this case, $S = \emptyset$. The only subset of the empty set is $\emptyset$. So S indeed has $2^0 = 1$ subset.

The inductive hypothesis is that for some n, S has 2^n subsets. Now, let $S' = \{x_1, x_2, \ldots, x_{n+1}\}$. If $T \subseteq S$, then T and $T \cup \{x_{n+1}\}$ are both subsets of S'. In fact, we can get all the subsets of S' by taking the subsets of S and adding or not adding x_{n+1} to each one. It is easy to see that we get all different subsets this way. Hence S' has exactly twice as many subsets as S. Since S had 2^n subsets, S' has 2^{n+1}. This completes the induction.

3. (10 pts) Prove the parallel summation identity: If m and n are nonnegative integers then

$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n}.$$

(Hint: Induction is easiest, although a proof by interpreting the two sides in terms of combinatorics is also possible.)

Here is the inductive argument. The base case is $n = 0$. In this case

$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m}{0} = 1$$
$$\binom{m+n+1}{n} = \binom{m+1}{0} = 1$$

which are indeed equal.

The inductive hypothesis is that

$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n}$$

for some $n \in \mathbb{Z}^{\geq 0}$. Now, for $n+1$, we have

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{m+k}{k} &= \sum_{k=0}^{n} \binom{m+k}{k} + \binom{m+n+1}{n+1} \\
&= \binom{m+n+1}{n} + \binom{m+n+1}{n+1} && \text{by the inductive hypothesis} \\
&= \binom{m+n+2}{n+1} && \text{by the Addition Identity.}
\end{aligned}$$

This is exactly what we wanted to show.

I will let you think about a combinatorial argument to prove this identity. Think about how you can form n-element subsets of an $(m+n+1)$-element set.

4. (10 pts) The Golden Dragon restaurant sells a three-item combo for \$5.99. The three-item combo includes a choice of three different main courses from twelve available, a choice of appetizer from hot and sour soup, egg drop soup, or spring rolls, and fried or steamed rice. To eat your main dishes, you can request either a plastic fork or a pair chopsticks. How many different three-item combos (including cutlery) can you pick up at the Golden Dragon?

First, we have $\binom{12}{3}$ ways to choose the three main dishes. Then we have 3 choices for the appetizer, 2 for the rice, and 2 for the cutlery. These choices are independent, so the total number of possible outcomes is

$$\binom{12}{3} \cdot 3 \cdot 2 \cdot 2 = 2640.$$

5. (10 pts) Prove the addition identity: If $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}$ then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

There are a few cases to consider:

Case $k < 0$: In this case both sides are 0

Case $k = 0$: In this case,

$$\binom{n}{k} = \binom{n}{0} = 1$$
$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n-1}{0} + \binom{n-1}{-1} = 1 + 0 = 1$$

Case $1 \le k \le n-1$: Now, we have

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= (n-1)! \left(\frac{n-k}{k!(n-k)(n-k-1)!} + \frac{k}{k(k-1)!(n-k)!} \right) \\
&= (n-1)! \frac{n-k+k}{k!(n-k)!} = \frac{n(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} = \binom{n}{k}
\end{aligned}$$

Case $k = n$: In this case,

$$\binom{n}{k} = \binom{n}{n} = 1$$
$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n-1}{n} + \binom{n-1}{n-1} = 0 + 1 = 1$$

Case $k > n$: In this case both sides are 0

6. (10 pts) **Extra credit problem.** In the game of Liar's Dice, each player rolls a set of dice (between 1 and 5 dice) concealed under a cup, looks at his/her own set and then makes a bid about all the dice in the game, including the other players' that (s)he cannot see. For example, suppose that you are one of four players and each player has five dice. When you look at your dice, you see $6, 4, 3, 1, 1$. You don't see your opponents dice at this point. You may bid that there are at least six 1s on the table. The next player–who sees only his/her own dice–can either accept your bid or challenge it. If (s)he chooses to challenge it, then all the dice are revealed and your bid is tested against them. If your bid is satisfied, the challenger loses a die, if it is not, you lose a die. Without going into all the intricacies of the game, let us suppose that there are four players, each with five dice. When you look at your dice, you see $6, 4, 3, 1, 1$ and decide to bid that there are at least six 1s on the table. What is the probability that your bid is satisfied? Remember that the probability is computed as the number of ways to achieve the desired outcome divided by the total number of possible outcomes. (Hint: The outcome of your own roll is not a probabilistic event for you as you already know what it is.)

   On a side note, I've read that the drinking game version of Liar's Dice is also called Mexicali. Not that I would know.

We already have two 1s. So the question is how likely we are to have at least four more 1s among the dice we cannot see. There are 15 such dice. Each could show any number between 1 and 6. So the total number of possible outcomes is 6^{15}. Notice that this calculation distinguishes between the different dice. To make this easier to visualize, imagine that the dice are all different colors.

Instead of counting the favorable outcomes, we will count the unfavorable ones. One unfavorable outcome is if there are no more 1s on the table. That is each die could show any number between 2 and 6. There are 5^{15} ways for this to happen.

Another unfavorable outcome is if there is only one more 1 on the table. This could be any one of the 15 dice, which gives us 15 choices. The remaining dice could show any number between 2 and 6 giving 5^{14} choices.

Yet another unfavorable outcome is if there are only two more 1s on the table. This could be any two of the 15 dice, which gives us $\binom{15}{2}$ choices. The remaining dice could show any number between 2 and 6, giving 5^{13} choices.

Finally, there could be only three more 1s on the table. This could be any three of the 15 dice, which gives us $\binom{15}{3}$ choices. The remaining dice could show any number between 2 and 6 giving 5^{12} choices.

Subtracting the number of unfavorable outcomes from all possible outcomes gives the number of favorable outcomes, which is

$$6^{15} - 5^{15} - 15(5^{14}) - \binom{15}{2}5^{13} - \binom{15}{3}5^{12}.$$

The probability of this happening is

$$\begin{aligned}
&\frac{6^{15} - 5^{15} - 15(5^{14}) - \binom{15}{2}5^{13} - \binom{15}{3}5^{12}}{6^{15}} = \\
&= 1 - \frac{5^{15} + 15(5^{14}) + \binom{15}{2}5^{13} + \binom{15}{3}5^{12}}{6^{15}} \\
&= 1 - \left(\frac{5}{6}\right)^{15} - \frac{15}{6}\left(\frac{5}{6}\right)^{14} - \frac{\binom{15}{2}}{6^2}\left(\frac{5}{6}\right)^{13} - \frac{\binom{15}{3}}{6^3}\left(\frac{5}{6}\right)^{12} \\
&= \frac{13607107447}{58773123072} \approx 0.23.
\end{aligned}$$

Here is another challenge for you. What is wrong with the following argument?

To have a favorable outcome, we need at least four more 1s on the table. These could be any four of the 15 dice, which gives us $\binom{15}{4}$ choices. The other 11 dice could show anything, giving another 6^{11} choices. So there are $\binom{15}{4}6^{11}$ favorable outcomes. Hence the probability of getting one of them is

$$\frac{\binom{15}{4}6^{11}}{6^{15}} = \frac{\binom{15}{4}}{6^4}.$$

Computing the number above with your calculator may give you a hint about the error in the argument.