Math 1101 Final Exam Solutions

May 10, 2006

1. (10 pts each)

(a) Use l'Hospital's rule to show that

$$\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = e^x$$

First, write

$$\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = \lim_{t\to\infty} e^{t\ln\left(1+\frac{x}{t}\right)}$$

We will find the limit of the exponent:

$$\lim_{t\to\infty}\left[t\ln\left(1+\frac{x}{t}\right)\right] = \lim_{t\to\infty}\frac{\ln\left(1+\frac{x}{t}\right)}{\frac{1}{t}}$$

Notice

$$t\to\infty \implies 1+\frac{x}{t}\to 1 \implies \ln\left(1+\frac{x}{t}\right)\to 0$$

and

$$t\to\infty \implies \frac{1}{t}\to 0$$

So this limit is a good candidate for l'Hospital's rule. Let's check that we can indeed use l'Hospital's rule.

$$\frac{d}{dt}\ln\left(1+\frac{x}{t}\right) = \frac{d}{dt}\ln\left(\frac{t+x}{t}\right) = \frac{d}{dt}\left(\ln(t+x)-\ln(t)\right) = \frac{1}{t+x}-\frac{1}{t}$$

$$\frac{d}{dt}\frac{1}{t} = -\frac{1}{t^2}$$

Both of these derivatives exist when t gets large (i.e. in a neighborhood of infinity) and $-1/t^2$ is not 0 there (although is approaches 0). So

$$\lim_{t\to\infty}\frac{\ln\left(1+\frac{x}{t}\right)}{\frac{1}{t}} = \lim_{t\to\infty}\frac{\frac{1}{t+x}-\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to\infty}\frac{-t^2}{1}\,\frac{t-(t+x)}{(t+x)t}$$

$$= \lim_{t\to\infty}\frac{tx}{t+x} = \lim_{t\to\infty}\frac{x}{1+\frac{x}{t}} = x$$

Since the exponential function is continuous everywhere (this is important!),

$$\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = \lim_{t\to\infty} e^{t\ln\left(1+\frac{x}{t}\right)} = e^x$$

(b) Use the definition of the derivative to show that $f(x)=|x|$ is not differentiable at 0.

$$f'(0) = \lim_{t\to 0}\frac{|t|-|0|}{t-0} = \lim_{t\to 0}\frac{|t|}{t}$$

Notice that $|t|/t$ behaves differently when $t<0$ and when $t>0$. So we'll look at the one-sided limits.

$$\lim_{t\to 0^+}\frac{|t|}{t} = \lim_{t\to 0^+}\frac{t}{t} = \lim_{t\to 0^+} 1 = 1$$

$$\lim_{t\to 0^-}\frac{|t|}{t} = \lim_{t\to 0^-}\frac{-t}{t} = \lim_{t\to 0^-} -1 = -1$$

Since the left and the right limits are different, the two-sided limit does not exist. Hence $f(x) = |x|$ is not differentiable at 0.

2. (a) (10 pts) Find the equation of the tangent line to $f(x) = \ln|\cos x|$ at $x = \pi$. (Hint: think about what the absolute value really means.)

As always, $|\cos x|$ is either $\cos x$ or $-\cos x$. Near $x = \pi$, $\cos x$ is close to -1, so $|\cos x| = -\cos x$ there.

$$f'(x) = \frac{d}{dx}\ln(-\cos(x)) = \frac{1}{-\cos(x)}\sin(x) = -\frac{\sin(x)}{\cos(x)} = -\tan(x)$$
$$f'(\pi) = -\tan(\pi) = 0$$

This will be the slope of the line. Since

$$f(\pi) = \ln|\cos\pi| = \ln|-1| = \ln(1) = 0$$

the tangent line also goes through the point $(\pi, 0)$. Hence the equation of the tangent line is $y = 0(x - \pi) + 0 = 0$.

(b) (10 pts) Find $y'(x)$ at the point $(1, 1)$ on the implicit curve

$$\arctan\left(\frac{y}{x}\right) = x^2 + y^2 + \frac{\pi}{4}$$

Using implicit differentiation with y a function of x:

$$\frac{d}{dx}\arctan\left(\frac{y}{x}\right) = \frac{d}{dx}\left(x^2 + y^2 + \frac{\pi}{4}\right)$$
$$\frac{1}{1 + \left(\frac{y}{x}\right)^2}\frac{d}{dx}\frac{y}{x} = 2x + 2yy'$$
$$\frac{x^2}{x^2 + y^2}\frac{y'x - y}{x^2} = 2x + 2yy'$$
$$\frac{y'x - y}{x^2 + y^2} = 2x + 2yy'$$

We can now substitute $y = x = 1$ and solve for y'

$$\frac{y' - 1}{2} = 2 + 2y'$$
$$-\frac{5}{2} = \frac{3}{2}y'$$
$$y' = -\frac{5}{3}$$

(c) (5 pts) Use the Fundamental Theorem of Calculus to evaluate

$$\int_1^e \frac{1}{x}\,dx$$

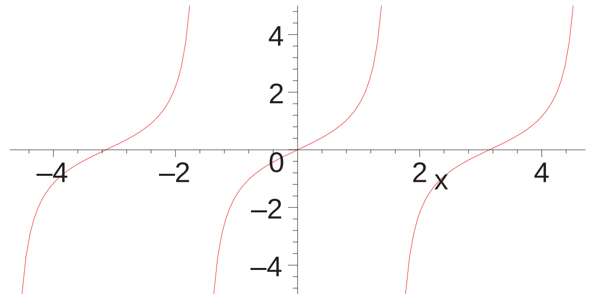
Since $\frac{d}{dx}\ln(x) = 1/x$, the FTC says

$$\int_1^e \frac{1}{x}\,dx = \ln(e) - \ln(1) = 1 - 0 = 1$$

3. (10 pts) In this problem, we will approximate $\tan(6°)$.

(a) Before you do any computation, sketch the graph of tan. What does this tell you about the value you expect for $\tan(6°)$?

Looking at the graph, we can tell that $\tan(6°)$ will be a positive number close to 0.



(b) Let $f(x) = \tan(x)$. Use linear approximation at $x = 0$ to estimate the value of $\tan(6°)$. Notice that the angle is in degrees, not radians. To convert it to radians, you may use $\pi \approx 3$, since this is only an approximation anyway.

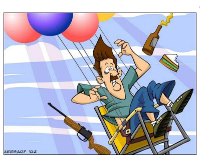
$$f(x) \approx f(0) + f'(0)(x - 0) = \tan(0) + \sec^2(0)x = x$$

But this only works if x is in radians, so

$$6° = \frac{6}{180}\pi = \frac{\pi}{30} \approx \frac{3}{30} = 0.1$$

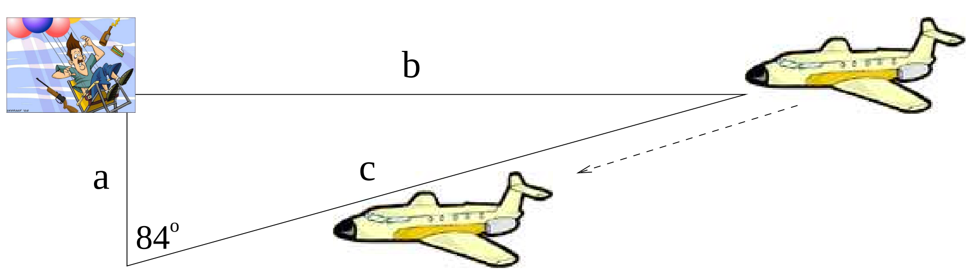
Now using the linear approximation

$$\tan(6°) \approx \tan(0.1) \approx 0.1$$



4. (15 pts) One fine summer day in 1982, Larry Walters fulfilled his life-long dream of flying by tying 45 weather balloons to his lawn chair and launching into the sky above the Los Angeles area. His plan was to rise to a modest altitude (perhaps a few hundred feet) and ride the gentle ocean breeze out to the desert. Instead, he climbed much higher and drifted into the path of air traffic landing at Long Beach Airport. Soon, the crew of a TWA passenger jet reported a guy floating in a lawn chair at 4800 m (16000 ft!) to air traffic control.
Let's say the TWA jet is flying at 500 km/h as it is descending on a 6° slope. When it is at the same altitude as Larry, it is 5 km from Larry. Its flight path takes it right under Larry.

(a) Draw a picture of the situation.



(b) How many meters below Larry will the jet pass? (Hint: Use the result from problem 3.)

As you can see from the picture,

$$\frac{a}{b} = \tan(6°) \implies a = b\tan(6°) \approx 0.1(5000\,\text{m}) = 500\,\text{m}$$

(c) At the time when Larry and the plane are at the same altitude (4800 m), how fast is the distance changing between them in km/min? You don't have to do the arithmetic with the numbers. (Hint: This is very similar to an exercise you had on the homework about an airplane and a control tower. Notice that in the triangle formed by the line between Larry and plane, the plane's path, and a vertical line through Larry, one angle is constant. Use the Law of Cosines.)

Notice that as the plane moves, the triangle in the picture will deform. (In particular, it will not remain a right-angle triangle.) But the angle at the bottom will always remain 84°. Using the Law of Cosines

$$b^2 = a^2 + c^2 - 2ac\cos(84^\circ)$$

We will now differentiate with respect to time. Note that a does not change with time:

$$\begin{aligned} 2b\frac{db}{dt} &= 2c\frac{dc}{dt} - 2a\cos(84^\circ)\frac{dc}{dt} \\ \frac{db}{dt} &= \frac{c - a\cos(84^\circ)}{b}\frac{dc}{dt} \end{aligned}$$

Using the picture above

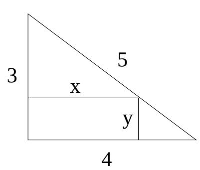
$$c = \sqrt{5^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\left(\frac{10}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{10^2 + 1^2} = \frac{\sqrt{101}}{2}$$

$$\cos(84^\circ) = \frac{b}{c} \approx \frac{5}{\frac{\sqrt{101}}{2}} = \frac{10}{\sqrt{101}}$$

We also know $dc/dt = -500/60\,\text{km/min}$ (yes, it's decreasing). So

$$\begin{aligned} \frac{db}{dt} &\approx \frac{\frac{\sqrt{101}}{2} - \frac{1}{2}\frac{10}{\sqrt{101}}}{5}\left(-\frac{50}{6}\right) = -\left(\frac{\sqrt{101}}{10} - \frac{1}{\sqrt{101}}\right)\frac{50}{6} \\ &= -\frac{101 - 10}{10\sqrt{101}}\frac{50}{6} = -\frac{455}{6\sqrt{101}} \end{aligned}$$

This is in km/min.



5. (15 pts) Consider the right-angle triangle whose sides are 3, 4, and 5 with an inscribed rectangle whose sides are to be parallel to the perpendicular sides of the triangle. (See picture.)

(a) Find the area of the rectangle as a function of x.

Using similar triangles in the picture:

$$\frac{y}{3} = \frac{4-x}{4} \implies y = 3 - \frac{3}{4}x$$

So

$$A(x) = x\left(3 - \frac{3}{4}x\right) = 3x - \frac{3}{4}x^2$$

(b) What are the possible values of x?

$$0 \le x \le 4$$

(c) Find the dimensions of the rectangle of largest area that can be inscribed in the triangle this way.

We are maximizing A over a closed interval. We know the maximum will be either at a critical point or at one of the endpoints. Since A is a polynomial, it is differentiable everywhere and the only kind of critical point is

$$0 = A'(x) = 3 - \frac{3}{2}x \implies x = 2$$

Since $A(0) = A(4) = 0$ and $A(2) = 3$, the maximum is at $x = 2$. So the dimensions of the maximal rectangle are $x = 2$ and $y = 3/2$.

(d) How do you know these dimensions really maximize the area of the rectangle? Do you need to test either the 1st or the 2nd derivative to be certain? Why or why not?

There is no need for either test. We know that on a closed interval, the maximum is either at a critical point or at an endpoint. We compared the values of A at all such points.

6. (15 pts)

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Use Rolle's Theorem to show that if f has two distinct roots, then f' has at least one root. (Don't assume f is a polynomial!)

Let $a < b$ be two roots of f. Then $f(a) = f(b) = 0$. Since f is differentiable, it is continuous. In particular, it is differentiable on (a, b) and continuous on $[a, b]$. So by Rolle's Theorem, there exists $x \in (a, b)$ such that $f'(x) = 0$, that is x is a root of f'.

(b) Now show that if $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function with three distinct roots, then f'' has at least one root. (Hint: you may use your result from the previous part.)

First, let $a < b < c$ be roots of f. Applying the result from part (a) at a, b, we get a root $x \in (a, b)$ of f', and applying it at b, c, we get a root $y \in (b, c)$ of f'. Notice that $x < b$ and $b < y$, so $x < y$. Since f' still satisfies the conditions in part (a), we get a root $z \in (x, y)$ of f''.

7. (20 pts) **Extra credit problem.** Don't attempt this problem until you are done with everything else.

A fixed point of the function f is some element c in the domain of f for which $f(c) = c$.

(a) Use the Intermediate Value Theorem to show that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. (Hint: consider the function $g(x) = f(x) - x$.)

If $f(0) = 0$ or $f(1) = 1$, then f already has a fixed point. So let's assume neither is the case. Let $g(x) = f(x) - x$. Since $f(0) \in [0, 1]$,

$$f(0) \neq 0 \implies f(0) > 0 \implies g(0) = f(0) - 0 > 0$$

Similarly, $f(1) \in [0, 1]$, so

$$f(1) \neq 1 \implies f(1) < 1 \implies g(1) = f(1) - 1 < 0$$

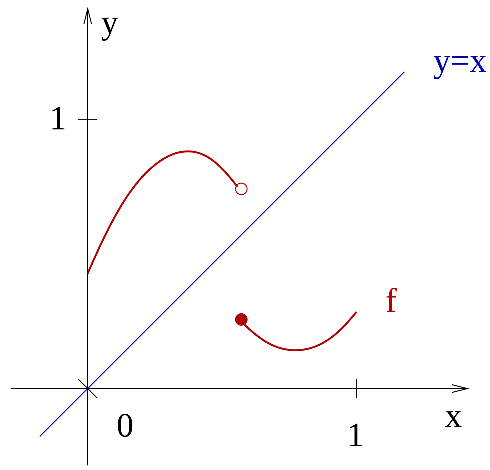
We know f is continuous, so $g(x) = f(x) - x$ is continuous. By the IVT, there must exist a point $c \in [0, 1]$ such that $g(c) = 0$. But

$$0 = g(c) = f(c) - c \implies f(c) = c$$

Remark: Fixed points are quite an interesting topic. For example, they appear in chaos theory. A generalization of this statement to functions on the surface of a sphere is sometimes called the Hairy Ball Theorem. It has important real life applications, such as it implies that at any time, there is a point on the surface of the earth where the wind is calm (meteorology) and you cannot comb a hedgehog without leaving some of its spikes pointing up punk style (zoology, hairdressing science).

(b) Find a function $f : [0, 1] \to [0, 1]$ which has no fixed point. (Hint: Such a function must be discontinuous by the previous part.)

Notice that a function whose graph crosses the line $y = x$ has a fixed point. (Why?) So all we need to do is find a function that does not cross this line, such as the function on the right. Try drawing a continuous function $[0, 1] \to [0, 1]$ that doesn't cross the line. Why can't you?



(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) \neq 1$ for any x. Use the Mean Value Theorem to show that f has at most one fixed point.

Suppose f had two fixed points a and b. Then $f(a) = a$ and $f(b) = b$. Since f is differentiable, it is continuous. In particular, it is differentiable on (a, b) and continuous on $[a, b]$. By the MVT, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1$$

But this would contradict $f'(x) \neq 1$.